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Abstract

In this thesis, a pseudodifferential calculus for a degenerate hyperbolic Cauchy problem is developed. The model for this problem originates from a certain observation in fluid mechanics, and is then extended to a more general class of hyperbolic Cauchy problems where the coefficients degenerate like a power of $t + |x|^2$ as $(t, x) \longrightarrow (0, 0)$.

Symbol classes and pseudodifferential operators are introduced. In this process, it becomes apparent that exactly in the origin, these operators are of type $(1, 1)$. Although these operators are not L^2-continuous in general, a proof of continuity in $\mathscr{C}([0, T], L^2(\mathbb{R}^d))$ is given for a suitable subclass.

An adapted scale of function spaces is defined, where at $t = 0$ these spaces coincide with 2-microlocal Sobolev spaces with respect to the Lagrangian $T_0^*\mathbb{R}^d$. In these spaces, energy estimates are derived, so that a symbolic approach can be applied to prove wellposedness of the Cauchy problem.

Zusammenfassung

Ziel dieser Arbeit ist es einen pseudodifferentiellen Kalkül zur Untersuchung eines degenerierten hyperbolischen Cauchy-Problems zu entwickeln. Das Modell für dieses Cauchy-Problem entstammt einer Beobachtung aus der Strömungsmechanik und wird anschließend zu allgemeineren hyperbolischen Cauchy-Problemen weiterentwickelt, deren Koeffizienten wie eine Potenz von $t+|x|^2$ degenerieren, wenn $(t,x) \longrightarrow (0,0)$.

Es werden Symbolklassen eingeführt und die entsprechenden Pseudodifferentialoperatoren definiert. Dabei stellt sich heraus, dass diese Operatoren im Ursprung vom Typ $(1,1)$ sind. Obgleich diese im Allgemeinen nicht L^2-stetig sind, gelingt unter zusätzlichen Annahmen der Beweis der Stetigkeit in $\mathscr{C}([0,T], L^2(\mathbb{R}^d))$ für eine spezielle Unterklasse von Symbolen.

Eine Skale von angepassten Funktionenräumen wird definiert, wobei an $t = 0$ diese Räume mit 2-mikrolokalen Sobolev-Räumen bezüglich des Lagrangian $T_0^*\mathbb{R}^d$ zusammenfallen. Mit Hilfe des symbolischen Zuganges werden für eine Lösung des Cauchy-Problems bezüglich dieser Räume Energie-Abschätzungen hergeleitet. Diese a priori-Abschätzungen werden genutzt um die Wohlgestelltheit des Cauchy-Problems zu beweisen.

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Contents

1 Introduction

> *Calvin: "You know, I don't think math is a science. I think it's a religion."*
> *Hobbes: "A religion?"*
> *Calvin: "Yeah. All these equations are like miracles. You take two numbers and when you add them, they magically become one new number! No one can say how it happens. You either believe it or you don't."*
>
> **— Comic by cartoonist Bill Watterson**

This thesis presents a study of the Cauchy problem for weakly hyperbolic systems of the form

$$\begin{cases} D_t U = A(t,x,D_x)U + F(t,x), & \text{in } (0,T)\times\mathbb{R}^d, \\ U(0,x) = U_0(x), & \text{on } \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $A(t,x,D_x)$ is an $N\times N$ first-order pseudodifferential operator. We will state the precise assumptions on the symbol $A(t,x,\xi)$ below.

Hyperbolic partial differential equations appear in various branches of physics in which conservation laws and finite-speed propagation are involved. The most basic hyperbolic PDE – modeling the vibration of a string in one space dimension – is the wave equation. It goes back to the work of d'Alembert in the 18^{th} century and is closely related to the transport equation. Another important linear hyperbolic system is the Maxwell system of electromagnetism. In theoretical physics, several semilinear equations arise, for example the Yang-Mills equations, the Maxwell system for polarized media or the Klein-Gordon equations. The nonlinear models – in most cases quasilinear – are even more numerous, for instance the Euler equations of gas dynamics, which are closely related to the theory of shock waves. Hyperbolicity is associated with a space-time reference frame. This means that one coordinate, namely physical time, plays a special role compared to the spatial coordinates. The analysis of hyperbolic PDEs uses a diversity of mathematical tools, ranging from microlocal analysis over pseudo- and paradifferential calculus to algebraic geometry. In this thesis, we will use techniques from pseudodifferential operators and microlocal analysis.

The theory of strictly hyperbolic equation is wellstudied. In our setting we assume that the pseudodifferential operator A is also strictly hyperbolic away from the crucial point $(t, x) = (0, 0)$, but degenerates like powers of $t + |x|^2$ as $(t, x) \longrightarrow (0, 0)$. Operators which exhibit such degeneracies are an example of weakly hyperbolic operators. A typical form of weakly hyperbolic equation reads as

$$u_{tt} - a(t)u_{xx} = 0, \;\; u(0, x) = u_0(x), \;\; u_t(0, x) = u_1(x),$$

as mentioned in [**DR1**]. If this equation is strictly hyperbolic in the sense that $a(t) \geq c > 0$, we have correctness in the classes of Sobolev spaces, and an energy estimate for a $\log$-Lipschitz coefficient $a(t)$, see [**CL**]. However, in the weakly hyperbolic theory, which means $a(t) \geq 0$, it turns out that $\mathscr{C}^\infty$-regularity of $a(t)$ is not sufficient for wellposedness of the Cauchy problem even in the classes of distributions. In [**CS**], Colombini and Spagnolo showed that for any $T > 0$ a coefficient $a(t) \in \mathscr{C}^\infty([0, \infty))$ and $\mathscr{C}^\infty$ data u_0, u_1 exist, such that the solution belongs to $\mathscr{C}^\infty([0, T), \mathscr{C}^\infty(\mathbb{R}))$, but not to $\mathscr{C}([0, T], \mathscr{D}'(\mathbb{R}))$. This function $a(t)$ is positive for $t < T$, oscillating for $t \to T - 0$, and identically zero for $t \geq T$.

For weakly hyperbolic equations another phenomenon arises, namely the loss of Sobolev regularity, even if the coefficients are smooth and have no oscillations. Consider for example the Cauchy problem

$$u_{tt} - t^2 u_{xx} = a u_x, \;\; u(0, x) = \varphi(x), \;\; u_t(0, x) = 0.$$

Then in [**Qi**] it is shown that the solution can be represented in the form

$$u(t, x) = \sum_{k=0}^{n} c_k t^{2k} \partial_x^k \varphi \left(x + \frac{t^2}{2} \right)$$

if $a = 4n + 1$, $n \in \mathbb{N}_0$. In particular, this means that

$$\varphi \in H^s \quad \Longrightarrow \quad u(t, \cdot) \in H^{s - \frac{a-1}{4}}.$$

In [**DW1**], Dreher and Witt studied this phenomenon for a more general higher-order operator of the form

$$P = \sum_{j+|\alpha| \leq m} a_{j\alpha}(t, x) t^{(j + (l_* + 1)|\alpha| - m)^+} D_t^j D_x^\alpha,$$

with coefficient $a_{j\alpha}(t, x)$ being smooth up to $t = 0$. After reducing the scalar equation into a system like (1.1), they used a symbolic approach to describe and analyze the corresponding pseudodifferential operator $A(t, x, D_x)$ and its properties, for example mapping properties on Sobolev spaces. After deriving energy estimates, they were able to prove existence and uniqueness to the Cauchy problem (1.1). In this thesis we will adopt this strategy.

For further results on loss of regularity, the propagation of singularities or, in general, degenerate hyperbolic Cauchy problems we refer to [**AN**; **DR2**; **Han**; **NU**; **Yag**; **AC1**; **AC2**].

Our prototypical operator originates from an observation on steady isentropic compressible flows, which are described by the Euler equations. If the flow is supersonic, which is that its velocity is greater than the speed of sound, the system of Euler equations is hyperbolic. If a supersonic line touches a sonic curve, this contact is generically of order two. After a hodograph transformation, this contact can be seen as multiplication with the factor $t + |x|^2$. Hence, our model equation reads as

$$(\partial_t^2 - (t + |x|^2)\Delta_x)u = f(t, x) \tag{1.2}$$

on $(0, T) \times \mathbb{R}^d$, and is weakly hyperbolic with degeneracy just in $(t, x) = (0, 0)$.

Developing a pseudodifferential calculus to study a certain kind of partial differential equations means to introduce a special class of symbols, which are closely related to the differential operator under consideration. In our case we study matrix-valued functions $a \in \mathscr{C}^\infty([0, T] \times \mathbb{R}^{2d}, M_{N\times N}(\mathbb{C}))$ with the property, that for each $(j, \alpha, \beta) \in \mathbb{N}_0^{1+2d}$, the estimate

$$|\partial_t^j \partial_x^\alpha \partial_\xi^\beta a(t, x, \xi)| \lesssim \langle\xi\rangle^{p+2j-|\beta|+|\alpha|} \langle\sigma\xi\rangle^{m-p-|\alpha|-2j}$$

holds for all $(t, x, \xi) \in [0, T] \times \mathbb{R}^{2d}$. We then say that a belongs to the symbol class $\Sigma^{m,p}$. Here, σ behaves like $\sqrt{t + |x|^2}$ near the origin. By Fourier transform we can define the corresponding class of pseudodifferential operators $\mathrm{Op}(\Sigma^{m,p})$. When passing to the origin it turns out that $a(0, \cdot, \cdot) \in S^p_{1,1}$, or, more generally, $(\partial_t^j a)(0, \cdot, \cdot) \in S^{p+2j}_{1,1}$ over $(x, \xi) = (0, 0)$. The analysis of operators of type $(1, 1)$ is technically difficult, because in general they are not L^2-continuous. This problem stem from the behavior of the twisted diagonal of the Fourier transform. However, it was a great achievement of Hörmander, see his book [**Hor5**], that if a is an operator of type (1,1) and the adjoint operator $\mathrm{Op}(a)^\dagger$ also belongs to $\mathrm{Op}(\Sigma^{m,p})$, then $\mathrm{Op}(a)$ is L^2-continuous. Other investigations on these operators were done by Johnsen, see [**Joh1**; **Joh2**]. Let $\mathrm{Op}(\Sigma^{m,p,\dagger})$ denote the class of all symbols for which the corresponding adjoint operators also belong to $\mathrm{Op}(\Sigma^{m,p})$. Then we are able to prove $\mathscr{C}([0, T], L^2(\mathbb{R}^d))$-continuity for operators belonging to $\mathrm{Op}(\Sigma^{0,0,\dagger})$. Actually this is not obvious, because we first have to show that, uniformly for every $t \in [0, T]$, the operator

$$\mathrm{Op}(a(t)) \colon \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$$

extends to a bounded one from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. The proof uses a pointwise application of the $T(1)$-theorem, as it is presented in [**Gra2**].

However, the class $\Sigma^{m,p}$ is not appropriate to define principal symbols. Therefore, we will refine that class, that is we have $a \in \widetilde{\Sigma}^{m,p}$ if a can be written in the form

$$a = \chi^+(a_0 + a_1) + a_r \tag{1.3}$$

where $a_j \in \Sigma^{(m-j),p}$ for $j = 0, 1$ and $a_r \in \Sigma^{m-2,p,\dagger} + \Sigma^{m-1,p-1,\dagger}$. Symbols in the class $\Sigma^{(m),p}$ admit a certain asymptotic expansion, and are homogeneous of order m in the covariable ξ. Moreover, the multiplication with the function χ^+ can be understood as a quantization $\Sigma^{(m),p} \longrightarrow \Sigma^{m,p}$. The main task is then to prove that the class of corresponding operators is closed under composition and taking adjoints. To prove this result, we have to estimate a certain type of oscillatory integral, again uniformly for all $t \in [0, T]$. Thanks to the special structure of the homogeneous component a_0 and its asymptotic expansion, we can apply similar techniques as in [**Kum**; **NR**]. With operators $A \in \mathrm{Op}(\widetilde{\Sigma}^{m,p})$ we then associate principal symbols $\sigma_\Psi^m(A)$ and $\sigma_{\Psi,d}^{m-1,p}(A)$, such that the short sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Op}(\Sigma^{m-2,p,\dagger}) + \mathrm{Op}(\Sigma^{m-1,p-1,\dagger}) \longrightarrow \mathrm{Op}(\widetilde{\Sigma}^{m,p}) \\ \xrightarrow{(\sigma_\Psi^m, \sigma_{\Psi,d}^{m-1,p})} \Sigma^{(m),p} \times \Sigma^{(m-1,p)} \longrightarrow 0 \end{aligned}$$

turns out to be exact. With this symbolic calculus for operators A at hand, coming from (1.3), we are able to argue on a purely algebraic level.

We also introduce function spaces $X^{s,\delta}$, to which a solution U to the system (1.1) belongs. Here, $s \in \mathbb{R}$ is the Sobolev regularity, while δ is related to the loss of regularity. At $t = 0$, these spaces coincide with a 2-microlocal Sobolev space with respect to the Lagrangian $T_0^*\mathbb{R}^d$. The concept of 2-microlocalization is basically microlocalization along a Lagrangian submanifold, see for example the famous paper of Bony, [**Bon**]. A characterization of the corresponding class of Sobolev spaces by use of Wavelets can be found in the book of Meyer, [**Mey**].

Finally, in order to make all of this useful for analysis, one needs mapping properties. We can prove that the class $\mathrm{Op}(\widetilde{\Sigma}^{m,p})$ maps $X^{s+m,\delta+p}$ continuously into $X^{s,\delta}$. This is done by a reduction argument to the class $\mathrm{Op}(\widetilde{\Sigma}^{0,0}) \subseteq \mathrm{Op}(\Sigma^{0,0,\dagger})$, which already contains continuous operators as proven before.

Our main result states as follows:

Theorem 1.1. *Let $A \in \mathrm{Op}(\widetilde{\Sigma}^{1,2})$. Assume there exists a $M_0 \in \Sigma^{(0),0}$ with $|\det M_0| \gtrsim 1$, $M_0 A_0 M_0^{-1}$ is Hermitian, such that*

$$I - 2x\left(M_{00} A_{00} M_{00}^{-1}\right)_\xi > 0,$$

where $M_{00} \in \Sigma^{(0,0)}$ is the $(0,0)$-bihomogeneous component of M_0.

Then for every M_{01} there exists a $\delta_0 = \delta_0(A_{00}, A_{01}) \in \mathbb{R}$ such that

$$\mathrm{Im}\left(\Phi(A_{00}, A_{01}, M_{00}, M_{01})\right) \leq \frac{\delta_0}{2r^2}\left(I - 2x\left(M_{00} A_{00} M_{00}^{-1}\right)_\xi\right), \tag{1.4}$$

and a $\gamma_0 = \gamma(A_{00})$, with the property that for all $s \in \mathbb{N}_0$, $\delta \geq \delta_0 + s\gamma_0$, $U_0 \in H^{s,\delta}$, $F \in Y^{s,\delta}$, the Cauchy problem (1.1) possesses a unique solution $U \in X^{s,\delta}$. Moreover, the a priori estimate

$$\|U\|_{X^{s,\delta}} \lesssim \|U_0\|_{H^{s,\delta}} + \|F\|_{Y^{s,\delta}}$$

is valid.

The precise formulation of Φ can be found in Chapter 4. The main idea is to derive an a priori estimate for a solution $U \in X^{s,\delta}$. Such estimates, often called energy estimates, give an upper bound of such a solution in terms of the source function F and the initial data U_0. We will show the validity of such energy estimates to our problem and derive existence and uniqueness by methods of functional analysis. Using this result, we apply this to the case of higher-order scalar equations. In particular, for our model problem we obtain the following result:

Theorem 1.2. *Let $(s,\delta) \in \mathbb{N}_0 \times \mathbb{R}_+$. Let $u_0 \in H^{s+1,\delta}(\mathbb{R}^d)$, $u_1 \in H^{s,\delta}(\mathbb{R}^d)$ and $f \in L^1 H^{s,\delta}$. Then the Cauchy problem*

$$\begin{cases} \partial_t^2 u - \sigma^2 \Delta_x u = f(t,x), & \textit{in } (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) & \textit{on } \mathbb{R}^d, \end{cases}$$

possesses an unique solution $u \in \mathscr{C} H^{s+1,\delta} \cap \mathscr{C}^1 H^{s,\delta}$.

This thesis is organized as follows: In Chapter 2 we briefly recall the system of Euler equations, describing compressible flows, and explain the connection to our model equation. We analyze its characteristics and derive a first energy estimate that shows the relation to the 2-microlocal Sobolev spaces.

In Chapter 3 we develop a pseudodifferential calculus that enables us to argue on a purely symbolic level. We study some examples and show continuity in $\mathscr{C}([0,T], L^2(\mathbb{R}^d))$ for a special subclass. After introducing the appropriate symbol class, we prove that the corresponding class of pseudodifferential operators is closed under compositions and taking adjoints. The rest of that Chapter is devoted to the principal symbols and function spaces.

In Chapter 4 we give the exact formulation of the main theorem and derive the basic energy estimate for operators $A \in \mathrm{Op}(\Sigma^{1,2,\dagger})$. Note, that at is point, no special structure on A is required. We reduce our system to an equivalent one, which has Hermitian principal and where the spectral parameter δ is shifted to zero. Use of an induction argument shows the validity of the energy estimate for all $s \in \mathbb{N}_0$ and corresponding δ. This Chapter is closed by an application to higher-order scalar and the model equation.

We end this thesis with an Appendix, where we recall basic concepts on oscillatory integrals, pseudodifferential operators of type $(1,1)$ and 2-microlocal Sobolev spaces.

2 Modeling the Problem

2.1 The model equation

We consider a two-dimensional steady isentropic compressible flow, which is described by the Euler equations of the form

$$\begin{aligned} (\rho u)_x + (\rho v)_y &= 0 \\ (\rho u^2 + p)_x + (\rho uv)_y &= 0 \\ (\rho uv)_x + (\rho v^2 + p)_y &= 0. \end{aligned} \tag{2.1}$$

Here $V = (u, v)$, p and ρ denote the velocity, pressure and density of the flow, respectively. For more information, see [**CF**; **And**]. Since the flow is isentropic, the pressure p is a smooth function depending on ρ, say $p = p(\rho)$, and we further assume $p'(\rho) > 0$.

Example 2.1. For a polytropic gas with adiabatic exponent $\gamma > 1$ we have

$$p(\rho) = \frac{1}{\gamma}\rho^\gamma.$$

Moreover, $q = |V| = \sqrt{u^2 + v^2}$ denotes the speed of the flow and the speed of sound c is defined as

$$c = \sqrt{p'(\rho)}.$$

The flow is said to be

- subsonic when $q < c$,
- sonic when $q = c$ and
- supersonic when $q > c$.

If the flow is supersonic, it turns out that the system (2.1) of equations is hyperbolic. Let now the hypersurface $\Sigma \subset \mathbb{R}^2$ describe the boundary of that problem. Then we impose boundary conditions on Σ of the form

$$(\rho, V)_{|\Sigma} = (\rho_0, V_0),$$

where we assume

$$\rho_0 > 0 \qquad \text{and} \qquad |V_0| > \sqrt{p'(\rho_0)}.$$

The latter means, that the flow is also supersonic up to the boundary.

To this problem the existence of short time solutions is well-known. However, in our setting we consider that these boundary conditions are violated, i.e. $q_0 = \sqrt{p'(\rho_0)}$, at one single point in Σ. Thus, we want to have an equation, that is strictly hyperbolic everywhere, except in one point. Also, we want to approach this degeneracy in a certain fashion.

Let $(\bar{\rho}, \bar{u}, \bar{v})$ be a solution to (2.1). Linearization around this solution has the principal symbol

$$\begin{pmatrix} \bar{u} & \bar{\rho} & 0 \\ \bar{u}^2 + p'(\bar{\rho}) & 2\bar{\rho}\bar{u} & 0 \\ \bar{u}\bar{v} & \bar{\rho}\bar{v} & \bar{\rho}\bar{u} \end{pmatrix} \xi + \begin{pmatrix} \bar{v} & 0 & \bar{\rho} \\ \bar{u}\bar{v} & \bar{\rho}\bar{v} & \bar{\rho}\bar{u} \\ \bar{v}^2 + p'(\bar{\rho}) & 0 & 2\bar{\rho}\bar{v} \end{pmatrix} \eta.$$

The determinant D with $\bar{c} = \sqrt{p'(\bar{\rho})} > 0$ is then given by

$$D = -(\bar{u}\xi + \bar{v}\eta)\left(\bar{c}^2(\xi^2 + \eta^2) - (\bar{u}\xi + \bar{v}\eta)^2\right)\bar{\rho}^2.$$

The factor $\bar{u}\xi + \bar{v}\eta$ correspond to planar waves of the form $F(\bar{v}x - \bar{u}y)$, i.e. $(\bar{u}\partial_x + \bar{v}\partial_y)[F(\bar{v}\xi - \bar{u}\eta)] = 0$, and will be ignored in the further investigations. However, the quadratic polynomial

$$(\xi, \eta) \longmapsto \bar{c}^2(\xi^2 + \eta^2) - (\bar{u}\xi + \bar{v}\eta)^2 \tag{2.2}$$

has discriminant $4\bar{c}^2(\bar{u}^2 + \bar{v}^2 - \bar{c}^2)$. Thus, if a supersonic curve touches a sonic line, this contact is generically of order two in the (u, v)-plane.

To see this contact in (x, y)-plane, we have to apply a hodograph transformation, which is a technique used to transform nonlinear partial differential equations into linear versions. The original form of a hodograph transformation is for a homogeneous quasi-linear system of two first-order equation for two known variables (u, v) in two independent variables (x, y). By regarding (x, y) as functions of (u, v) and assuming that the Jacobian does not vanisch nor is infinity, one can rewrite the system for the unknowns (x, y) in the variables (u, v). Basically we apply an interchanging of the dependent and independent variables in the equation to achieve linearity. Specifically, consider the system of two equations of the form,

$$\begin{pmatrix} u \\ v \end{pmatrix}_x + A(u, v, x, y) \begin{pmatrix} u \\ v \end{pmatrix}_y = 0, \tag{2.3}$$

where the coefficient A is

$$A(u, v, x, y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The two eigenvalues, denoted by $E_\pm$, satisfy

$$E_\pm^2 - (a_{11} + a_{22})E_\pm + \det(A) = 0.$$

We introduce the hodograph transformation $T\colon (x, y) \longmapsto (u, v)$. Then the system (2.3) reduces to

$$\begin{pmatrix} y_v \\ -y_u \end{pmatrix} + A(u, v, x, y) \begin{pmatrix} -x_v \\ x_u \end{pmatrix} = 0. \tag{2.4}$$

Its eigenvalues, denoted by $e_\pm$, satisfy

$$a_{12}e_\pm^2 - (a_{22} - a_{11})e_\pm - a_{21} = 0.$$

Then it turns out that a characteristic of (2.3) in the (x, y) plane is mapped into a characteristic of (2.4) in the (u, v) plan via T. For more details on these transformation, see [**Ber**; **CF**].

However, the relation on the contact of supersonic and sonic curves does not change. So in the original coordinates this degeneracy corresponds to multiplication with the factor $t + |x|^2$. Since the system is hyperbolic, we have a certain variable, which determines the hyperbolic direction. We will denote this variable by t. For more investigations on sonic curves in transonic and subsonic-sonic flows, see [**WX**].

Remark 2.2. If we further assume the flow to be irrotational, we can also introduce a velocity potential function $\phi = \phi(x, y)$, such that $\nabla\phi = V$. Then system (2.1) can be written as a nonlinear second-order equation of the form

$$(c^2 - \phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (c^2 - \phi_y^2)\phi_{yy} = 0, \tag{2.5}$$

see again [**CF**; **And**]. The potential flow equation can be understood as a combination of continuity, momentum and energy equation, and its advantage is the following: if one knows the potential ϕ, all other components (V, c, p, ρ) can be calculated. With $\partial_1 = \partial_x$, $\partial_2 = \partial_y$ equation (2.5) has the form

$$\sum_{i,j=1}^{2} a_{ij}\partial_i\partial_j\phi = 0$$

with coefficient-matrix

$$A = (a_{ij}) = \begin{pmatrix} c^2 - \phi_x^2 & -\phi_x\phi_y \\ -\phi_x\phi_y & c^2 - \phi_y^2 \end{pmatrix}.$$

The eigenvalues $\lambda_{1,2} = \lambda_{1,2}(|\nabla|)$ of A are given by the equation

$$\det(A - \lambda I) = (c^2 - \phi_x^2 - \lambda)(c^2 - \phi_y^2 - \lambda) - \phi_x^2\phi_y^2 = 0,$$

which has the solutions

$$\lambda_1 = c^2 \quad \text{and} \quad \lambda_2 = c^2 - |\nabla\phi|^2.$$

Since the flow is assumed to be supersonic, we have $\lambda_2 = c^2 - |\nabla\phi|^2 < 0$ and the discriminant of (2.2) can be represented as $-4\lambda_1\lambda_2$.

Thus, our model equation reads as

$$(\partial_t^2 - (t + x^2)\Delta_x)u = 0$$

for $(t, x) \in (0, T) \times \mathbb{R}^d$. For $d = 1$ the principal symbol of this equation is given by

$$p(t, x, \tau, \xi) = \tau^2 - (t + x^2)\xi^2 = (\tau - \sqrt{t + x^2}\xi)(\tau + \sqrt{t + x^2}\xi).$$

The each of the two real roots $\tau(t, x, \xi) = \pm\sqrt{t + x^2}\xi$ we obtain the characteristic initial value problem

$$\frac{dx}{dt} = x'(t) = \pm\sqrt{t + x^2(t)}, \qquad x(0) = 0. \tag{2.6}$$

Note that in our setting $x(0) = 0$ is the only possible initial value.

We now want to analyze the problem (2.6), say just for the positive sign. The function $F\colon [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$, $F(t, x) = \sqrt{t + x^2}$ is continuous and Lipschitz continuous with respect to the second variable. Thus, by Picard–Lindelöf's theorem there exists a unique characteristic $x(t)$. Using Picard's iteration scheme, we additionally obtain

$$x(t) = \frac{3}{2}t^{3/2} + O(t^{5/2}) \qquad \text{as } t \to 0+.$$

This means, that for small t we asymptotically have the same characteristic as for the Tricomi-operator $\partial_t^2 - t\partial_x^2$ in the hyperbolic region. We remark, that we have an similar characteristic for the negative root of the principal symbol.

By extending this to multi-dimensional space variable $x \in \mathbb{R}^d$, we are interested in the wellposedness of the Cauchy problem

$$\begin{cases} \partial_t^2 u - (t + |x|^2)\Delta_x u = f(t,x), & \text{in } (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) & \text{on } \mathbb{R}^d, \end{cases}$$

for a suitable source function f and initial data u_0 and u_1. Now, this second-order scalar equation can be generalized to the higher-dimensional case, that is

$$\begin{cases} Pu \qquad\quad = f(t,x), & \text{in } (0,T) \times \mathbb{R}^d, \\ (\partial_t^j u)(0,x) = u_0(x) & \text{on } \mathbb{R}^d,\ j = 0, \dots m-1, \end{cases}$$

for

$$P = D_t^m + \sum_{j=1}^{m} a_j(t,x,D_x) D_t^{m-j}.$$

Here, the a_j are pseudodifferential operators of order j that degenerate like a power of $t + |x|^2$ as $(t,x) \longrightarrow (0,0)$. Such higher-dimensional equation then can be reduced to a first order system of size $m \times m$ as

$$\begin{cases} D_t U \qquad = A(t,x,D_x)U + F(t,x), & \text{in } (0,T) \times \mathbb{R}^d, \\ U(0,x) = U_0(x), & \text{on } \mathbb{R}^d, \end{cases}$$

where $A(t,x,D_x)$ is again a pseudodifferential operator of order 1 with same degeneracy.

In Chapter 3 we will develop a pseudodifferential calculus, where we analyze these kinds of operators.

2.2 A first observation

When dealing with the equation itself, the following result arises:

Lemma 2.3. *Let $u_0, u_1 \in \mathscr{C}_c^\infty(\mathbb{R}^d)$, $\frac{u_1}{|x|} \in L^2(\mathbb{R}^d)$ and $f \in L^2((0,T) \times \mathbb{R}^d)$. A solution u to the Cauchy problem*

$$\begin{cases} \partial_t^2 u - (t + |x|^2)\Delta_x u = f(t,x), & \textit{in } (0,T) \times \mathbb{R}^d, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) & \textit{on } \mathbb{R}^d, \end{cases}$$

satisfies on $(0,T)$ *the energy inequality*

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \left\|\frac{u_t(t)}{\sqrt{t+|x|^2}}\right\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\int_0^t \left\|\frac{u_t(s)}{t+|x|^2}\right\|^2_{L^2(\mathbb{R}^d)} ds \le \|\nabla u_0\|^2_{L^2(\mathbb{R}^d)} + \left\|\frac{u_1}{|x|}\right\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|f(s)\|^2_{L^2(\mathbb{R}^d)} ds.$$

Proof. Write $r = \sqrt{t+|x|^2}$. We can show this inequality by direct formal computations. Since

$$\frac{d}{dt}\left(\frac{u_t^2}{r^2}\right) = \frac{2u_t u_{tt}}{r^2} - \left(\frac{u_t}{r^2}\right)^2, \tag{2.7}$$

we have

$$\begin{aligned}\frac{d}{dt}\left\|\frac{u_t}{r}\right\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \frac{d}{dt}\left(\frac{u_t}{r}\right)^2 dx = \int_{\mathbb{R}^d}\left[\frac{2u_t u_{tt}}{r^2} - \left(\frac{u_t}{r^2}\right)^2\right] dx \\ &= 2\left\langle \frac{u_{tt}}{r^2}, u_t\right\rangle_{L^2(\mathbb{R}^d)} - \left\|\frac{u_t}{r^2}\right\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Furthermore, we see

$$\begin{aligned}\frac{d}{dt}\|\nabla u\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\frac{d}{dt}|\nabla u|^2 dx = 2\int_{\mathbb{R}^d} \nabla u\cdot\nabla u_t dx \\ &= -2\int_{\mathbb{R}^d}\Delta u\cdot u_t dx = -2\langle \Delta u, u_t\rangle_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8) yields

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\frac{u_t}{r}\right\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u\|^2_{L^2(\mathbb{R}^d)}\right) + \left\|\frac{u_t}{r^2}\right\|^2_{L^2(\mathbb{R}^d)} \\ &= 2\left\langle\frac{u_{tt}}{r^2}, u_t\right\rangle_{L^2(\mathbb{R}^d)} - \left\|\frac{u_t}{r^2}\right\|^2_{L^2(\mathbb{R}^d)} - 2\langle\Delta u, u_t\rangle_{L^2(\mathbb{R}^d)} + \left\|\frac{u_t}{r^2}\right\|^2_{L^2(\mathbb{R}^d)} \\ &= 2\left\langle\frac{u_{tt}}{r^2} - \Delta u, u_t\right\rangle_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{2.9}$$

Since $u_{tt} - r^2\Delta u = f$ is equivalent to $\frac{u_{tt}}{r^2} - \Delta u = \frac{f}{r^2}$ on $(0,T)$ and in view of Young's weighted inequality, we get

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\frac{u_t}{r}\right\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u\|^2_{L^2(\mathbb{R}^d)}\right) + \left\|\frac{u_t}{r^2}\right\|^2_{L^2(\mathbb{R}^d)} \\ &= 2\left\langle f, \frac{u_t}{r^2}\right\rangle_{L^2(\mathbb{R}^d)} \le 2\|f(t)\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\left\|\frac{u_t}{r^2}\right\|^2_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{2.10}$$

Integrating (2.10) with respect to t yields

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \left\| \frac{u_t(t)}{r} \right\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\int_0^t \left\| \frac{u_t(s)}{r^2} \right\|^2_{L^2(\mathbb{R}^d)} ds$$
$$\leq \|\nabla u_0\|^2_{L^2(\mathbb{R}^d)} + \left\| \frac{u_1}{|x|} \right\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t \|f(s)\|^2_{L^2(\mathbb{R}^d)} ds.$$

This is the desired estimate.

□

Remark 2.4. Using the sequel notation of function spaces, this lemma would imply $u \in \mathscr{C}H^{1,1} \cap \mathscr{C}^1 H^{0,1}$. By setting $s = \delta = 0$ in Theorem 1.2, our theory will give

$$u \in \mathscr{C}H^{1,0} \cap \mathscr{C}^1 H^{0,0}$$

and thus provide a better result.

This inequality is just a first heuristical observation and will not play any role in the latter analysis. However, since we have to assume that the function $\frac{u_1}{|x|}$ is square-integrable, this shows a connection to the 2-microlocal spaces, see the Appendix for an introduction.

3 Pseudodifferential Calculus

Developing a pseudodifferential calculus to study a certain kind of partial differential equations means to introduce a special class of symbols, which are closely related to the differential operator of the given problem. In our setting we are not interested in long-time behavior, but in existence and uniqueness of solutions near the origin. Therefore, we define the function $\sigma\colon [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}$ via

$$\sigma(t,x) = \begin{cases} \sqrt{t+|x|^2}, & t+|x|^2 \le \frac{1}{2}, \\ 1, & t+|x|^2 \ge 1. \end{cases}$$

The function σ is positive for $(t,x)\neq(0,0)$ and belongs to the function space $\mathscr{C}^{\frac{1}{2}}([0,T]\times\mathbb{R}^d)\cap\mathscr{C}^\infty(([0,T]\times\mathbb{R}^d)\setminus\{(0,0)\})$ and describes the kind of singularity near the origin.

3.1 The symbol class $\Sigma^{m,p}$

We use σ as a weight function in the following symbol estimate.

Definition 3.1. For $(m,p)\in\mathbb{R}^2$, the symbol class $\Sigma^{m,p}$ consists of all functions $a\in\mathscr{C}^\infty([0,T]\times\mathbb{R}^{2d},M_{N\times N}(\mathbb{C}))$ such that for each multiindex $(j,\alpha,\beta)\in\mathbb{N}_0^{1+2d}$ the estimate

$$|\partial_t^j\partial_x^\alpha\partial_\xi^\beta a(t,x,\xi)| \lesssim \langle\xi\rangle^{p+2j-|\beta|+|\alpha|}\langle\sigma\xi\rangle^{m-p-|\alpha|-2j}$$

holds for all $(t,x,\xi)\in[0,T]\times\mathbb{R}^{2d}$.

For $\ell\in\mathbb{N}_0$ let us also define a system of semi-norms via

$$|a|_{m,p;\ell} = \sup_{\substack{(t,x,\xi)\in[0,T]\times\mathbb{R}^{2d} \\ j+|\alpha|+|\beta|\le\ell}} \langle\xi\rangle^{-p-2j+|\beta|-|\alpha|}\langle\sigma\xi\rangle^{-m+p+|\alpha|+2j}|\partial_t^j\partial_x^\alpha\partial_\xi^\beta a|.$$

It is not difficult to check, that $\Sigma^{m,p}$ together with these semi-norms forms a Fréchet space.

Away form $(t,x)=(0,0)$ the symbol class $\Sigma^{m,p}$ coincides with $S^m_{1,0}$. Moreover, by restricting to $t=0$, we obtain the 2-microlocal estimates in variables (x,ξ) with respect to the Lagrangian $T_0^*\mathbb{R}^d$. More precisely, we get symbols $a(0,x,\xi) \in \Sigma_0^{m,p}$, that is

$$|\partial_x^\alpha \partial_\xi^\beta a(0,x,\xi)| \lesssim \langle |x|\xi\rangle^{m-p-|\alpha|}\langle\xi\rangle^{p-|\beta|+|\alpha|}.$$

For more details on this clas, see the Appendix. Another difficulty arises directly in the origin $(t,x)=(0,0)$, since we then get the estimate

$$|\partial_x^\alpha \partial_\xi^\beta a(0,0,\xi)| \lesssim \langle\xi\rangle^{p-|\beta|+|\alpha|}.$$

Thus, we have $a(0,\cdot,\cdot) \in S^p_{1,1}$, or, more generally, $(\partial_t^j a)(0,\cdot,\cdot) \in S^{p+2j}_{1,1}$ over $(x,\xi) = (0,0)$. The analysis of operators of type $(1,1)$ is technically difficult, because in general they are not L^2-continuous. The problems stem from the behavior of the twisted diagonal of the Fourier transform. We will recall some results on operators of type $(1,1)$ in the Appendix. Furthermore, by definition, the weight functions $\langle\xi\rangle$ and $\langle\sigma\xi\rangle$ are symbols in $\Sigma^{1,1}$ and $\Sigma^{1,0}$, respectively. Moreover, we have the embedding

$$S^m_{1,0}([0,T]\times\mathbb{R}^{2d}) \subseteq \Sigma^{m,m},$$

which follows directly from the estimate

$$|\partial_t^j\partial_x^\alpha\partial_\xi^\beta a| \lesssim \langle\xi\rangle^{m-\beta} \lesssim \langle\sigma\xi\rangle^{-2j-|\alpha|}\langle\xi\rangle^{m+2j+|\alpha|-|\beta}$$

for an $a \in S^m_{1,0}([0,T]\times\mathbb{R}^{2d})$, since $\langle\sigma\xi\rangle \lesssim \langle\xi\rangle$.

Example 3.2. We also want to discuss the example of $a(t,x,\xi)=\sigma^2$ near the origin. It is $\sigma^2 \in \Sigma^{0,-2}$. Indeed, we have

$$\sigma^2 = \frac{1+|\xi|^2}{1+|\xi|^2}\sigma^2 = \frac{\sigma^2+\sigma^2|\xi|^2}{1+|\xi|^2} \leq \frac{1+\sigma^2|\xi|^2}{1+|\xi|^2} = \langle\sigma\xi\rangle^2\langle\xi\rangle^{-2}.$$

Moreover, we easily compute

$$\partial_t\sigma^2 = 1 \quad \text{and} \quad \partial_{x_i}\sigma^2 = 2x_i \lesssim \sigma \lesssim \frac{\langle\sigma\xi\rangle}{\langle\xi\rangle}.$$

All higher derivatives vanish.

For later use, we also set

$$\Sigma^{-\infty,p} := \bigcap_{m\in\mathbb{R}} \Sigma^{m,p} \quad \text{and} \quad \Sigma^{-\infty,-\infty} := \bigcap_{(m,p)\in\mathbb{R}^2} \Sigma^{m,p} = \mathscr{C}_b^\infty([0,T]\times\mathbb{R}^d_x, \mathscr{S}(\mathbb{R}^d_\xi)).$$

Let us prove our first result, an approximation lemma:

Lemma 3.3. *Let $a \in \Sigma^{0,0}$, $0 \le \varepsilon \le 1$ and set $a_\varepsilon = a(t,x,\varepsilon\xi)$. Then a_ε is bounded in $\Sigma^{0,0}$ and*

$$a_\varepsilon \xrightarrow{\Sigma^{m,p}} a_0$$

for all $p \ge m > 0$ as $\varepsilon \longrightarrow 0$.

Proof. We prove this with the additional assumption that $m, p \in (0,1]$, the general case follows immediately. Since $a_0 = a(t,x,0) \in \Sigma^{0,0}$, we are finished, if we can prove that for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^{2d}$ and all $(j,\alpha,\beta) \in \mathbb{N}_0^{1+2d}$ there exists a constant C satisfying

$$\langle\xi\rangle^{-p-2j-|\alpha|+|\beta|}\langle\sigma\xi\rangle^{-m+p+|\alpha|+2j}|\partial_t^j\partial_x^\alpha\partial_\xi^\beta(a(t,x,\varepsilon\xi)-a(t,x,0))| \le C\varepsilon^m.$$

Consider first $|\beta| \ge 1$. Then the left-hand side can be estimated by

$$\begin{aligned}
&\langle\xi\rangle^{-p-2j-|\alpha|+|\beta|}\langle\sigma\xi\rangle^{-m+p+|\alpha|+2j}|\partial_t^j\partial_x^\alpha\partial_\xi^\beta(a(t,x,\varepsilon\xi)-a(t,x,0))| \\
&\qquad \le C\langle\xi\rangle^{-p-2j-|\alpha|+|\beta|}\langle\sigma\xi\rangle^{-m+p+|\alpha|+2j}\varepsilon^{|\beta|}\langle\varepsilon\xi\rangle^{2j-|\beta|+|\alpha|}\langle\varepsilon\sigma\xi\rangle^{-|\alpha|-2j} \\
&\qquad = C\left(\frac{\langle\varepsilon\xi\rangle}{\langle\xi\rangle}\right)^{2j+|\alpha|}\langle\xi\rangle^{-p+|\beta|}\langle\varepsilon\xi\rangle^{-|\beta|}\langle\sigma\xi\rangle^{-m+p}\left(\frac{\langle\sigma\xi\rangle}{\langle\varepsilon\sigma\xi\rangle}\right)^{2j+|\alpha|}\varepsilon^{|\beta|}.
\end{aligned}$$

Note that

$$\frac{\langle\varepsilon\xi\rangle\langle\sigma\xi\rangle}{\langle\xi\rangle\langle\varepsilon\sigma\xi\rangle} \le 1.$$

This leads to

$$\begin{aligned}
&\langle\xi\rangle^{-p-2j-|\alpha|+|\beta|}\langle\sigma\xi\rangle^{-m+p+|\alpha|+2j}|\partial_t^j\partial_x^\alpha\partial_\xi^\beta(a(t,x,\varepsilon\xi)-a(t,x,0))| \\
&\qquad \le C\varepsilon^{|\beta|}\langle\xi\rangle^{-p+|\beta|}\langle\varepsilon\xi\rangle^{-|\beta|}\langle\sigma\xi\rangle^{-m+p}.
\end{aligned}$$

and further, by algebraic manipulations,

$$\begin{aligned}
&\langle\xi\rangle^{-p-2j-|\alpha|+|\beta|}\langle\sigma\xi\rangle^{-m+p+|\alpha|+2j}|\partial_t^j\partial_x^\alpha\partial_\xi^\beta(a(t,x,\varepsilon\xi)-a(t,x,0))| \\
&\qquad \le C\varepsilon^{|\beta|}\left(\frac{\langle\sigma\xi\rangle}{\langle\xi\rangle}\right)^{p-m}\langle\xi\rangle^{|\beta|-m}\varepsilon^m\varepsilon^{-m}\langle\varepsilon\xi\rangle^{-|\beta|} \\
&\qquad \le C\varepsilon^m\left(\frac{\varepsilon\langle\xi\rangle}{\langle\varepsilon\xi\rangle}\right)^{|\beta|-m}\langle\varepsilon\xi\rangle^{-m} \\
&\qquad \le C\varepsilon^m.
\end{aligned}$$

Consider now $\beta = 0$. By using Taylor's formula, we obtain

$$(\partial_t^j\partial_x^\alpha a)(t,x,\varepsilon\xi) - (\partial_t^j\partial_x^\alpha a)(t,x,0) = \sum_{|\beta|=1}\frac{(\varepsilon\xi)^\beta\partial_t^j\partial_x^\alpha\partial_\xi^\beta a(t,x,\vartheta\varepsilon\xi)}{\beta!}$$

for a certain $\vartheta \in [0,1]$. Estimating the right side brings us to

$$\begin{aligned}|(\partial_t^j \partial_x^\alpha a)(t,x,\varepsilon\xi) - (\partial_t^j \partial_x^\alpha a)(t,x,0)| &\leq \sum_{i=1}^d |\varepsilon\xi| \cdot |\partial_t^j \partial_x^\alpha \partial_{\xi_i} a(t,x,\vartheta\varepsilon\xi)| \\ &\leq \varepsilon\langle\xi\rangle\langle\vartheta\varepsilon\xi\rangle^{2j-1+|\alpha|}\langle\vartheta\varepsilon\sigma\xi\rangle^{-|\alpha|-2j}.\end{aligned}$$

By similar arguments as in the previous case, we get

$$\begin{aligned}&\langle\xi\rangle^{-p-2j-|\alpha|+|\beta|}\langle\sigma\xi\rangle^{-m+p+|\alpha|+2j}|\partial_t^j\partial_x^\alpha\partial_\xi^\beta(a(t,x,\varepsilon\xi)-a(t,x,0))| \\ &\qquad \leq C\varepsilon\langle\xi\rangle\langle\xi\rangle^{-p}\langle\vartheta\varepsilon\xi\rangle^{-1}\langle\sigma\xi\rangle^{-m+p} \\ &\qquad \leq C\varepsilon^m\left(\frac{\varepsilon\langle\xi\rangle}{\langle\vartheta\varepsilon\xi\rangle}\right)^{1-m} \leq C\varepsilon^m.\end{aligned}$$

This completes the proof. □

We are now going to introduce the corresponding pseudodifferential operators, namely the class $\mathrm{Op}(\Sigma^{m,p})$.

Theorem 3.4. *If $a \in \Sigma^{m,p}$ and $u \in \mathscr{C}^\infty([0,T],\mathscr{S}(\mathbb{R}^d))$, then*

$$\mathrm{Op}(a)u := a(t,x,D_x)u(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\xi}a(t,x,\xi)\hat{u}(t,\xi)d\xi$$

defines a function $a(t,x,D_x)u \in \mathscr{C}^\infty([0,T],\mathscr{S}(\mathbb{R}^d))$ and the bilinear map

$$\Sigma^{m,p}\times\mathscr{C}^\infty([0,T],\mathscr{S}(\mathbb{R}^d)) \longrightarrow \mathscr{C}^\infty([0,T],\mathscr{S}(\mathbb{R}^d)), \quad (a,u)\longmapsto a(t,x,D_x)u$$

is continuous. Moreover $[D_t,\mathrm{Op}(a)]u = \mathrm{Op}(D_t a)$.

Proof. For every fixed $t \in [0,T]$ we have $\hat{u}(t,\xi) \in \mathscr{S}(\mathbb{R}^d)$ and the function

$$(\mathrm{Op}(a)u)(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\xi}a(t,x,\xi)\hat{u}(t,\xi)d\xi$$

is continuous. Moreover,

$$|(\mathrm{Op}(a)u)(t,x)| \lesssim \int_{\mathbb{R}^d}\langle\xi\rangle^p\langle\sigma\xi\rangle^{m-p}|\hat{u}(t,\xi)|d\xi \cdot \sup_{(t,x,\xi)\in[0,T]\times\mathbb{R}^{2d}}|a(t,x,\xi)|\langle\xi\rangle^{-p}\langle\sigma\xi\rangle^{p-m},$$

which shows, that $(\mathrm{Op}(a)u)(t,\cdot)$ belongs to $\mathscr{S}(\mathbb{R}^d)$. Since a and $\hat{u}$ depend smoothly on the variable t, one also gets that $\mathrm{Op}(a)u \in \mathscr{C}^\infty([0,T],\mathscr{S}(\mathbb{R}^d))$. Moreover,

$$\begin{aligned}
D_t \,\mathrm{Op}(a)u &= D_t \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\xi} a(t,x,\xi)\hat{u}(t,\xi)d\xi \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\xi}(D_t a)(t,x,\xi)\hat{u}(t,\xi) + a(t,x,\xi)(D_t\hat{u})(t,\xi))d\xi \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\xi}(D_t a)(t,x,\xi)\hat{u}(t,\xi)d\xi \\
&\qquad + \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\xi} a(t,x,\xi)(D_t\hat{u})(t,\xi))d\xi \\
&= \mathrm{Op}(D_t a) + \mathrm{Op}(a)D_t u,
\end{aligned}$$

which is exactly $[D_t, \mathrm{Op}(a)]u = \mathrm{Op}(D_t a)$. □

Definition 3.5. We call $a(t,x,D_x)$ a pseudodifferential operator of order m and bi-order (m,p) and set

$$\mathrm{Op}(\Sigma^{m,p}) := \{\mathrm{Op}(a) \,|\, a \in \Sigma^{m,p}\}.$$

For later use, we also define $\Sigma^{m,p,\dagger} := \{a \in \Sigma^{m,p} \,|\, \mathrm{Op}(a)^\dagger \in \mathrm{Op}(\Sigma^{m,p})\}$, where $\dagger$ denotes the operation of taking adjoint with respect to L^2.

We now want to prove the following lemma:

Lemma 3.6. *Let $\chi \in \mathscr{C}^\infty(\mathbb{R})$ with $\chi(z) = 0$ if $|z| \leq 1/2$ and $\chi(z) = 1$ for $|z| \geq 1$ be an excision function. Then $\chi^+(t,x,\xi) := \chi(\sigma\langle\xi\rangle)$ belongs to $\Sigma^{0,0}$ while*

$$\chi^-(t,x,\xi) := 1 - \chi(\sigma\langle\xi\rangle) \in \Sigma^{-\infty,0}.$$

Proof. By definition, we have $|\chi^+(t,x,\xi)| \lesssim 1$. For all $n \in \mathbb{N}$, the functions

$$\chi^{(n)}(z) = \frac{d^n}{dz^n}\chi$$

are bounded and compactly supported with same support. So now fix an multiindex $(j,\alpha,\beta) \in \mathbb{N}_0^{1+2d}$ with $j + |\alpha| + |\beta| \geq 1$. By higher multi-dimensional chain rule for composed function the function $(\partial_t^j \partial_x^\alpha \partial_\xi^\beta \chi^+)(t,x,\xi)$ can be written as a sum, where each summand is basically a product of derivatives of $\chi(z)$ and derivatives of $\sigma\langle\xi\rangle$ up to combinatorial constants. Since all derivatives $\chi^{(n)}(z)$ are zero outside the set $\{1/2 \leq |z| \leq 1\}$, also $(\partial_t^j \partial_x^\alpha \partial_\xi^\beta \chi)(\sigma\langle\xi\rangle)$ vanishes outside $\{1/2 \leq \sigma\langle\xi\rangle \leq 1\}$. Hence, we can find a constant $C_{j\alpha\beta}$ such that

$$|\partial_t^j \partial_x^\alpha \partial_\xi^\beta \chi^+| \leq C_{j\alpha\beta}\langle\xi\rangle^{2j-|\beta|+|\alpha|}\langle\sigma\xi\rangle^{-|\alpha|-2j}.$$

This means that $\chi^+ \in \Sigma^{0,0}$.

Since χ^- is compactly supported, we immediately get

$$|\chi^-(t,x,\xi)| \lesssim \langle\sigma\xi\rangle^m \qquad \text{for all } m \in \mathbb{R}.$$

Similar arguments as above give us $\chi^- \in \Sigma^{-\infty,0}$. $\square$

In the next proposition we list properties of the symbol classes $\Sigma^{m,p}$.

Proposition 3.7.

(i) Let $a \in \Sigma^{m,p}$ and $(j,\alpha,\beta) \in \mathbb{N}_0^{1+2d}$, then $\partial_t^j \partial_x^\alpha \partial_\xi^\beta a \in \Sigma^{m-|\beta|,p+2j+|\alpha|-|\beta|}$.

(ii) For $(m,p),(m',p') \in \mathbb{R}^2$ the composition $\Sigma^{m,p} \cdot \Sigma^{m',p'} \subseteq \Sigma^{m+m',p+p'}$ holds.

(iii) We have the embedding $\Sigma^{m,p} \subseteq \Sigma^{m',p'} \iff m \le m'$ and $p \le p'$.

Proof. Statement (i) is proven by definition of the symbol class, (ii) by a calculation using to product rule for derivatives.

Let us now prove (iii)and first $\Sigma^{m,p} \subseteq \Sigma^{m',p'}$. Then we get that

$$\langle\sigma\xi\rangle^{m-p}\langle\xi\rangle^p \lesssim \langle\sigma\xi\rangle^{m'-p'}\langle\xi\rangle^{p'}$$

holds for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^{2d}$. For the particular point $(0,0,\xi)$ we have $\langle\xi\rangle^p \lesssim \langle\xi\rangle^{p'}$ and since $\langle\xi\rangle \ge 1$ for all $\xi \in \mathbb{R}^d$, we finally get $p \le p'$. By setting $|\xi| = 1$ we arrive at $\langle\sigma\rangle^{m-p} \lesssim \langle\sigma\rangle^{m'-p'}$. Since this is true for all $(t,x) \in [0,T] \times \mathbb{R}^d$, we conclude $m - p \le m' - p'$. Putting these two conditions together, we get $m \le m'$ and $p \le p'$.

Conversely let $m \le m'$ and $p \le p'$. Observe that

$$\frac{\langle\xi\rangle}{\langle\sigma\xi\rangle} \ge 1 \implies \left(\frac{\langle\xi\rangle}{\langle\sigma\xi\rangle}\right)^p \ge \left(\frac{\langle\xi\rangle}{\langle\sigma\xi\rangle}\right)^{p'} \implies \langle\xi\rangle^p\langle\sigma\xi\rangle^{-p} \le \langle\xi\rangle^{p'}\langle\sigma\xi\rangle^{-p'}.$$

Moreover, $\langle\sigma\xi\rangle^m \le \langle\sigma\xi\rangle^{m'}$ and so

$$\langle\xi\rangle^p\langle\sigma\xi\rangle^{m-p} \le \langle\xi\rangle^{p'}\langle\sigma\xi\rangle^{m'-p'}.$$

This proves $\Sigma^{m,p} \subseteq \Sigma^{m',p'}$. $\square$

Next, for the sake of completeness, we want to prove asymptotic completeness.

Proposition 3.8. *Let $a_k \in \Sigma^{m_k,p}$, $k \in \mathbb{N}$, be an arbitrary sequence, where m_k is monotone decreasing with $m_k \longrightarrow -\infty$ as $k \longrightarrow \infty$. Then there is a symbol $a \in \Sigma^{m,p}$ with $m := m_1$ such that for every M, there is an $N(M)$ so that for all $N \geq N(M)$:*

$$a(t,x,\xi) - \sum_{k=0}^{N} a_k(t,x,\xi) \in \Sigma^{m-M,p}.$$

The element $a \in \Sigma^{m,p}$ is uniquely determined by this property modulo $\Sigma^{-\infty,p}$.

Definition 3.9. We call any such a asymptotic sum of the a_k, $k \in \mathbb{N}$, and write

$$a(t,x,\xi) \sim \sum_{k\in\mathbb{N}} a_k(t,x,\xi).$$

Proof. In order to apply Theorem A.20, let us set $E^k := \Sigma^{m_k,p}$ for $k \in \mathbb{N}$, with the natural system of semi-norms

$$|a|_{m_k,p;\ell} := \sup_{\substack{(t,x,\xi)\in[0,T]\times\mathbb{R}^{2d} \\ j+|\alpha|+|\beta|\leq l}} \langle\xi\rangle^{-p-2j+|\beta|-|\alpha|} \langle\sigma\xi\rangle^{-m_k+p+|\alpha|+2j} |\partial_t^j \partial_x^\alpha \partial_\xi^\beta a|.$$

Let again

$$\chi(z) = \begin{cases} 0, & |z| \leq 1/2 \\ 1, & |z| \geq 1 \end{cases}$$

be an excision function and define $\chi^k(c)\colon \Sigma^{m_k,p} \longrightarrow \Sigma^{m_k,p}$, $c \in \mathbb{R}_+$, as the operator of multiplication by $\chi(c^{-1}\sigma\langle\xi\rangle)$. Then it is clear, that the diagram

$$\begin{array}{ccc} \Sigma^{m_{k+1},p} & \hookrightarrow & \Sigma^{m_k,p} \\ {\scriptstyle \chi(c^{-1}\sigma\langle\xi\rangle)}\downarrow & & \downarrow{\scriptstyle \chi(c^{-1}\sigma\langle\xi\rangle)} \\ \Sigma^{m_{k+1},p} & \hookrightarrow & \Sigma^{m_k,p} \end{array}$$

commutes. By previous results, we have $\chi^- \in \Sigma^{-\infty,0}$ and so we get

$$(1 - \chi(c^{-1}\sigma\langle\xi\rangle))a \in \Sigma^{-\infty,p}$$

for all $a \in \Sigma^{m_k,p}$, $k \in \mathbb{N}$. It suffices now to check, that for every $k \in \mathbb{N}$ and all $(j,\alpha,\beta) \in \mathbb{N}_0^{1+2d}$ there is an index i, such that $b(t,x,\xi) \in \Sigma^{\mu,p}$ for $\mu \leq m_i$ implies

$$\sup_{(t,x,\xi)\in[0,T]\times\mathbb{R}^{2d}} \langle\xi\rangle^{-p-2j+|\beta|-|\alpha|} \langle\sigma\xi\rangle^{-m_k+p+|\alpha|+2j} |\partial_t^j \partial_x^\alpha \partial_\xi^\beta \{\chi(c^{-1}\sigma\langle\xi\rangle) b(t,x,\xi)\}| \longrightarrow 0$$

as $c \longrightarrow \infty$. Let us show this for arbitray $\mu < m_k$. We have

$$\begin{aligned}
&\sup \langle\xi\rangle^{-p-2j+|\beta|-|\alpha|}\langle\sigma\xi\rangle^{-m_k+p+|\alpha|+2j}|\partial_t^j\partial_x^\alpha\partial_\xi^\beta\{\chi(c^{-1}\sigma\langle\xi\rangle)b(t,x,\xi)\}| \\
=&\sup \langle\xi\rangle^{-p-2j+|\beta|-|\alpha|}\langle\sigma\xi\rangle^{\mu-m_k}\langle\sigma\xi\rangle^{-\mu+p+|\alpha|+2j}|\partial_t^j\partial_x^\alpha\partial_\xi^\beta\{\chi(c^{-1}\sigma\langle\xi\rangle)b(t,x,\xi)\}|
\end{aligned}$$

with supremum taken over $(t, x, \xi) \in [0, T] \times \mathbb{R}^{2d}$. Since $\chi(z)$ is an excision function, we have $\chi(z) = 0$ for $|z| < R$ with some $R > 0$. Then $\chi(c^{-1}\sigma\langle\xi\rangle)$ vanishes for $|c^{-1}\sigma\langle\xi\rangle| < R$. Thus, it is permitted to take the supremum of $|\sigma\langle\xi\rangle| \geq cR$, and so

$$\sup \langle\xi\rangle^{-p-2j+|\beta|-|\alpha|}\langle\sigma\xi\rangle^{-m_k+p+|\alpha|+2j}|\partial_t^j\partial_x^\alpha\partial_\xi^\beta\{\chi(c^{-1}\sigma\langle\xi\rangle)b(t,x,\xi)\}| \leq K(c)\langle cR\rangle^{\mu-m_k}$$

with

$$K(c) = \sup \langle\xi\rangle^{-p-2j+|\beta|-|\alpha|}\langle\sigma\xi\rangle^{-\mu+p+|\alpha|+2j}|\partial_t^j\partial_x^\alpha\partial_\xi^\beta\{\chi(c^{-1}\sigma\langle\xi\rangle)b(t,x,\xi)\}|.$$

Now, $K(c)$ is uniformly bounded for $c \geq \varepsilon$ for all $\varepsilon > 0$. So there exists an L with $K(c) \leq L$ for c sufficently large, for which

$$|\chi(c^{-1}\sigma\langle\xi\rangle)b|_{m_k,p,\ell} \leq \frac{L}{\langle cR\rangle^{m_k-\mu}}.$$

Since $m_k - \mu > 0$, we get

$$|(\chi(c^{-1}\sigma\langle\xi\rangle)b|_{m_k,p,\ell} \longrightarrow 0 \qquad \text{as } c \longrightarrow \infty.$$

In view of Theorem A.20, there now exists a sequence of constants $c_k \in \mathbb{R}_+$, such that

$$a(t,x,\xi) = \sum_{k=1}^{\infty}\chi(c_k^{-1}\sigma\langle\xi\rangle)a_k(t,x,\xi)$$

converges in $\Sigma^{m,p}$, and has the property

$$a(t,x,\xi) - \sum_{k=1}^{N}a_k(t,x,\xi) \in \Sigma^{m_{N+1},p}$$

for all $N \in \mathbb{N}$. Furthermore, a is unique modulo $\Sigma^{-\infty,p}$ and so

$$a(t,x,\xi) \sim \sum_{k\in\mathbb{N}}a_k(t,x,\xi).$$

This completes the proof. $\square$

3.2 $\mathscr{C}([0,T], L^2(\mathbb{R}^d))$-continuity for $\mathrm{Op}(\Sigma^{0,0,\dagger})$

In this section we want to show, that the operator $A := \mathrm{Op}(a)$ is $\mathscr{C}([0,T], L^2(\mathbb{R}^d))$-bounded, provided $a \in \Sigma^{0,0,\dagger}$.

The strategy is to consider A as a linear operator on $\mathscr{C}([0,T], \mathscr{S}(\mathbb{R}^d))$, which can be represented in the its usual form

$$(Au)(t,x) = \int_{\mathbb{R}^d} K(t,x,y)u(t,y)\,dy, \tag{3.1}$$

where K is the t-dependent Schwartz kernel of A and satisfies a certain kernel estimate. This inequality guarantees K to be a standard kernel. An application of the $T(1)$-theorem shows, that it extends pointwise to $L^2(\mathbb{R}^d)$, uniformly in $t \in [0,T]$. See the Appendix for an introduction to standard kernels and the $T(1)$-theorem. This result will be the basis for us to prove mapping properties between function spaces.

Lemma 3.10. *Suppose $a \in \Sigma^{0,0,\dagger}$. Then for every $t \in [0,T]$ the pseudodifferential operator*

$$A(t) = a(t,x,D_x) \colon \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$$

can be written in the form as in (3.1) with time-dependent kernel

$$K_t(x,y) = k(t,x,x-y)$$

where $k(t,x,z)$ is smooth away from $z = 0$ and satisfies

$$|\partial_t^j \partial_x^\alpha \partial_z^\beta k(t,x,z)| \lesssim |z|^{-d-|\alpha|-|\beta|-2j}$$

for all $(j,\alpha,\beta) \in \mathbb{N}_0^{1+2d}$.

Proof. In this proof we use the common technique of Littlewood-Paley decomposition, see for instance [**Abe**; **Bon**; **Hor3**]. A standard calculation first shows, that

$$\begin{aligned}(Au)(t,x) &= \int_{\mathbb{R}^d} a(t,x,\xi)e^{ix\xi}\hat{u}(t,\xi)\,d\xi \\ &= \int_{\mathbb{R}^{2d}} a(t,x,\xi)e^{i(x-y)\xi}u(t,y)\,d\xi\,dy \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} a(t,x,\xi)e^{i(x-y)\xi}\,d\xi \right) u(t,y)\,dy.\end{aligned}$$

We therefore have a representation of desired form with

$$K_t(x,y) = k(t,x,x-y) := \int_{\mathbb{R}^d} a(t,x,\xi) e^{i(x-y)\xi}\, d\xi,$$

namely as an oscillatory integral. It is clear, that k is smooth away from $z=0$. Now let $\phi \in \mathscr{C}^\infty(\mathbb{R}^d)$, such that $\phi \equiv 1$ in $|\xi| \le 1$ and $\operatorname{supp}\phi \subseteq \{|\xi| \le 2\}$. The Littlewood-Paley decomposition is then given by functions $\phi_\ell(\xi) := \phi(2^{-\ell}\xi) - \phi(2^{1-\ell}\xi)$ with $\operatorname{supp}\phi_\ell$ contained in the set $\{2^{\ell-1} \le |\xi| \le 2^{\ell+1}\}$. It follows, that

$$\phi(\xi) + \sum_{\ell=1}^{\infty} \phi_\ell(\xi) = \lim_{\ell\to\infty} \phi(2^{-\ell}\xi) \equiv 1.$$

For all those functions ϕ_ℓ we obtain the estimate $|\partial_\xi^\beta \phi_\ell(\xi)| \lesssim \langle\xi\rangle^{-|\beta|}$, $\beta \in \mathbb{N}_0^d$, which mean, that $\phi_\ell \in \Sigma^{0,0,\dagger}$ for all $\ell \in \mathbb{N}_0$. Since the symbol a is in $\Sigma^{0,0,\dagger}$, we get, that

$$a_\ell(t,x,\xi) = a(t,x,\xi)\phi_\ell(\xi) \quad \text{and} \quad a_0(t,x,\xi) = a(t,x,\xi)\phi(\xi)$$

form a family out of functions again in $\widetilde{\Sigma}^{0,0,\dagger}$. Hence, by quantization, the operator A can be represented as

$$A(t) = a(t,x,D_x) = \sum_{\ell=0}^{\infty} A_{t,\ell} = \sum_{\ell=0}^{\infty} a_\ell(t,x,D_x),$$

where $A_{t,\ell}$ possesses the Schwartz kernel

$$k_\ell(t,x,z) = \int_{\mathbb{R}^d} a_\ell(t,x,\xi) e^{i\xi z}\, d\xi.$$

Since the integrand functions a_ℓ are compactly supported in ξ, we easily compute the derivatives

$$\partial_t^j \partial_x^\alpha \partial_z^\beta k_\ell(t,x,z) = \int_{\mathbb{R}^d} \partial_t^j \partial_x^\alpha a_\ell(t,x,\xi)(i\xi)^\beta e^{i\xi x}\, d\xi.$$

Multiplication on the left side with $(iz)^\gamma$, $\gamma \in \mathbb{N}_0^d$, leads to derivatives ∂_ξ^γ in the integral. An integration by part yields

$$(iz)^\gamma \partial_t^j \partial_x^\alpha \partial_z^\beta k_\ell(t,x,z) = \int_{\mathbb{R}^d} \partial_\xi^\gamma((i\xi)^\beta \partial_t^j \partial_x^\alpha a_\ell(t,x,\xi)) e^{i\xi x}\, d\xi.$$

The task is now to find an upper bound for the integrand. Indeed, by using the symbol estimates, we get

$$\begin{aligned}
|\partial_\xi^\gamma((i\xi)^\beta \partial_t^j \partial_x^\alpha a_\ell(t,x,\xi))| &\lesssim \langle\xi\rangle^{|\alpha|+|\beta|-|\gamma|+2j} \langle\sigma\xi\rangle^{-|\alpha|-2j} \\
&\le \langle\xi\rangle^{|\alpha|+|\beta|-|\gamma|+2j} \\
&\lesssim 2^{\ell(|\alpha|+|\beta|-|\gamma|+2j)}.
\end{aligned}$$

Since the support is of size $2^{\ell d}$, we get

$$|\partial_t^j \partial_x^\alpha \partial_z^\beta k_\ell(t,x,z)| \lesssim 2^{\ell(d+|\alpha|+|\beta|-M+2j)}|z|^{-M}$$

for every $M \in \mathbb{N}_0$ where $M := |\gamma|$.

Consider now $|z| \geq 1$. Then, by an application of the geometric series, we see

$$\begin{aligned}
|\partial_t^j \partial_x^\alpha \partial_z^\beta k(t,x,z)| &\leq \sum_{\ell=0}^{\infty} |\partial_t^j \partial_x^\alpha \partial_z^\beta k_\ell(t,x,z)| \\
&\lesssim |z|^{-M} \sum_{\ell=0}^{\infty} 2^{\ell(d+|\alpha|+|\beta|-M+2j)} \\
&\lesssim |z|^{-M} \lesssim |z|^{-d-|\alpha|-|\beta|-2j},
\end{aligned}$$

provided $d + |\alpha| + |\beta| - M + 2j > 0$, that is $M > d + |\alpha| + |\beta| + 2j$.

For the case $|z| < 1$ we split the kernel into two different sums, say

$$\begin{aligned}
|\partial_t^j \partial_x^\alpha \partial_z^\beta k(t,x,z)| &\leq \sum_{\ell=0}^{\infty} |\partial_t^j \partial_x^\alpha \partial_z^\beta k_\ell(t,x,z)| \\
&\lesssim \sum_{2^\ell < |z|^{-1}} |\partial_t^j \partial_x^\alpha \partial_z^\beta k_\ell(t,x,z)| + \sum_{2^\ell \geq |z|^{-1}} |\partial_t^j \partial_x^\alpha \partial_z^\beta k_\ell(t,x,z)|.
\end{aligned}$$

For given $|z| < 1$, there are just finitely many ℓ, such that $2^\ell < |z|^{-1}$. So the first sum is finite. By setting $M = 0$, and noting that $d + |\alpha| + |\beta| + 2j > 0$, we obtain

$$\sum_{2^\ell < |z|^{-1}} |\partial_t^j \partial_x^\alpha \partial_z^\beta k_\ell(t,x,z)| \leq \sum_{2^\ell < |z|^{-1}} 2^{\ell(d+|\alpha|+|\beta|+2j)} \lesssim |z|^{-d-|\alpha|-|\beta|-2j}.$$

In a similar fashion to the first case we get for the remainder sum

$$\begin{aligned}
\sum_{2^\ell \geq |z|^{-1}} |\partial_t^j \partial_x^\alpha \partial_z^\beta k_\ell(t,x,z)| &\lesssim |z|^{-M} \sum_{\ell=0}^{\infty} 2^{\ell(d+|\alpha|+|\beta|-M+2j)} \\
&\lesssim |z|^{-M} \lesssim |z|^{-d-|\alpha|-|\beta|-2j},
\end{aligned}$$

if $M > d + |\alpha| + |\beta| + 2j$. This finishes the proof of the lemma. □

Remark 3.11. Considering $A(t) = a(t,x,D_x)$ as a time-dependent family of operators $\{a(t,x,D_x)\}_{t\in[0,T]}$, we have $A(t) \in \mathrm{Op}(S^0_{1,0})$ for every $t \in (0,T]$, but not uniformly in t. For $t > 0$ we would get the better estimate

$$|\partial_t^j \partial_x^\alpha \partial_z^\beta k(t,x,z)| \leq C_{j\alpha\beta}(t)|z|^{-d-|\beta|-2j}$$

with $C_{j\alpha\beta}(t) \longrightarrow \infty$ as $t \longrightarrow 0$. However, the estimate in the previous lemma holds for every $t \in [0,T]$.

Using that lemma, we obviously get in particular $|K_t(x,y)| \lesssim |x-y|^{-d}$ for all $x, y \in \mathbb{R}^d$ with $x \neq y$ and uniformly in $t \in [0,T]$. Moreover, by the kernel estimates it holds

$$|\partial_{x_i} K_t(x,y)| \lesssim |x-y|^{-d-1} \quad \text{and} \quad |\partial_{y_i} K_t(x,y)| \lesssim |x-y|^{-d-1}$$

respectively, for all $x \neq y$. Putting these statements together yields

$$|\nabla_x K_t(x,y)| + |\nabla_y K_t(x,y)| \lesssim |x-y|^{-d-1}$$

for all $x, y \in \mathbb{R}^d$, $x \neq y$, and uniformly in t. So $K_t(x,y)$ defines a family of standard kernels, depending on $t \in [0,T]$. We are now ready to prove the following proposition.

Proposition 3.12. *Let $a \in \Sigma^{0,0,\dagger}$. Then, for every $t \in [0,T]$, the corresponding linear operator*

$$A(t) = a(t,x,D_x)\colon \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$$

extends to a bounded operator $A(t)\colon L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$.

Proof. First, we recover the symbol of the pseudodifferential operator A. By defining $f(t,x) := e^{ix\theta}$ with parameter $\theta \in \mathbb{R}^d$, we get $\hat{f}(t,\xi) = \delta(\xi - \theta)$. Using the integral representation of A, we obtain

$$A(e^{i\theta\cdot})(t,x) = \int_{\mathbb{R}^d} a(t,x,\xi) e^{ix\xi} \delta(\xi - \theta)\, d\xi = a(t,x,\theta) e^{ix\theta}.$$

For every $\theta \in \mathbb{R}^d$ and pointwise for all $t \in [0,T]$ we have, that the function $a(t,x,\theta)e^{ix\theta}$ is bounded, since

$$|a(t,x,\theta)e^{ix\theta}| = |a(t,x,\theta)| \lesssim 1,$$

because $a \in \Sigma^{0,0,\dagger}$. In particular, we get $A(e^{i\theta\cdot})(t,\cdot) \in L^\infty(\mathbb{R}^d) \subset \mathrm{BMO}(\mathbb{R}^d)$, see Lemma A.22. With a similar argument we also obtain $A^\dagger(e^{i\theta\cdot})(t,\cdot) \in L^\infty(\mathbb{R}^d) \subset \mathrm{BMO}(\mathbb{R}^d)$. Since the upper bounds do not depend on θ, we conclude, that

$$\sup_{\theta \in \mathbb{R}^d} \|A(e^{i\theta\cdot})(t,\cdot)\|_{\mathrm{BMO}} + \sup_{\theta \in \mathbb{R}^d} \|A^*(e^{i\theta\cdot})(t,\cdot)\|_{\mathrm{BMO}} < \infty,$$

uniformly for all $t \in [0,T]$. By Proposition A.30 (iii), the operator $A(t)$ extends to a bounded linear map $L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$, uniformly in $t \in [0,T]$. $\square$

Let us finally prove two important Corollaries:

Corollary 3.13. *Let $a \in \Sigma^{0,0,\dagger}$. Then $\mathrm{Op}(a)$ is continuous from $L^2((0,T)\times\mathbb{R}^d)$ to $L^2((0,T)\times\mathbb{R}^d)$.*

Proof. Let $v \in L^2(\mathbb{R}^d)$. From the previous proposition we have

$$\|A(t)v\|_{L^2(\mathbb{R}^d)} \lesssim \|v\|_{L^2(\mathbb{R}^d)} \qquad \text{uniformly in } t \in [0,T].$$

Identifying a function $u \in L^2((0,T) \times \mathbb{R}^d)$ as a family $\{u(t)\}_{t\in[0,T]} \subseteq L^2(\mathbb{R}^d)$ we see, that

$$\|A(t)u(t,\cdot)\|_{L^2(\mathbb{R}^d)} \lesssim \|u(t,\cdot)\|_{L^2(\mathbb{R}^d)} \qquad \text{uniformly in } t \in [0,T].$$

Squaring and integrating over t brings us to

$$\begin{aligned} \|Au\|^2_{L^2((0,T)\times\mathbb{R}^d)} &= \int_0^T \|A(t)u(t)\|^2_{L^2(\mathbb{R}^d)}\,dt \\ &\lesssim \int_0^T \|u(t)\|^2_{L^2(\mathbb{R}^d)}\,dt = \|u\|^2_{L^2((0,T)\times\mathbb{R}^d)} \end{aligned}$$

and so $\|Au\|_{L^2((0,T)\times\mathbb{R}^d)} \lesssim \|u\|_{L^2((0,T)\times\mathbb{R}^d)}$. This proves the L^2-continuity of $\mathrm{Op}(a)$. □

Corollary 3.14. *Let $a \in \Sigma^{0,0,\dagger}$. Then $\mathrm{Op}(a)$ is continuous from $\mathscr{C}([0,T], L^2(\mathbb{R}^d))$ to $\mathscr{C}([0,T], L^2(\mathbb{R}^d))$.*

Proof. We have

$$\begin{aligned} \|A(t)u(t) - A(0)u(0)\|_{L^2(\mathbb{R}^d)} &= \|A(t)u(t) - A(t)u(0) + A(t)u(0) - A(0)u(0)\|_{L^2(\mathbb{R}^d)} \\ &\leq \|A(t)\|_{\mathscr{L}(L^2(\mathbb{R}^d))}\|u(t) - u(0)\|_{L^2(\mathbb{R}^d)} \\ &\quad + \|A(t) - A(0)\|_{\mathscr{L}(L^2(\mathbb{R}^d))}\|u(0)\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

This shows $Au \in \mathscr{C}([0,T], L^2(\mathbb{R}^d))$. The continuity of the map A follows immediately. □

3.3 The symbol class $\widetilde{\Sigma}^{m,p}$

The class $\Sigma^{m,p}$ is not appropriate for defining principal symbols. Therefore, we have to refine the space $\Sigma^{m,p}$. This is done by the construction, presented in the sequel.

Let $\Upsilon = ([0,T] \times \mathbb{R}^d) \setminus \{(0,0)\}$ and

$$\Theta = \{(\widetilde{t}, \widetilde{x}) \in [0,1] \times \mathbb{R}^d \,|\, \widetilde{t} + |\widetilde{x}|^2 = 1\}.$$

We introduce generalized polar coordinates $(r, \vartheta) \in \mathbb{R}_+ \times \Theta$, where $\vartheta = (\widetilde{t}, \widetilde{x})$, in such a manner that $t = r^2\widetilde{t}$ and $x = r\widetilde{x}$. In particular, we get

$$t + |x|^2 = r^2(\widetilde{t} + |\widetilde{x}|^2) = r^2,$$

this is $r = \sqrt{t + |x|^2}$, since $r \in \mathbb{R}_+$.

Definition 3.15. The class $\Sigma^{(m),p}$, $(m,p) \in \mathbb{R}^2$, consists of all amplitude functions $a \in \mathscr{C}^\infty(\Upsilon, S^{(m)}(\dot{\mathbb{R}}^d))$, which admit an asymptotic expansion of the form

$$a(t,x,\xi) \sim \sum_{k \geq 0} \sigma^{m-p+k} b_k(\vartheta, \xi) \quad \text{as } r \to +0$$

with $b_k \in \mathscr{C}^\infty(\Theta, S^{(m)}(\dot{\mathbb{R}}^d))$. This means, that for any $K \in \mathbb{N}_0$ and all multiindices $(j,\alpha,\beta) \in \mathbb{N}_0^{1+2d}$ the estimate

$$\left| \partial_t^j \partial_x^\alpha \partial_\xi^\beta \left(a(t,x,\xi) - \sum_{0 \leq k < K} \sigma^{m-p+k} b_k(\vartheta,\xi) \right) \right| \lesssim \sigma^{m-p-2j-|\alpha|+K} |\xi|^{m-|\beta|}$$

holds.

Note that for $a \in \Sigma^{(m),p}$ we have the symbol estimate (set $K = 0$)

$$|\partial_t^j \partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)| \lesssim \sigma^{m-p-2j-|\alpha|} |\xi|^{m-|\beta|}.$$

Lemma 3.16. *We have the following scaling in* $\Sigma^{(m),p}$*:*

$$\frac{1}{\lambda^p} a(\lambda^{-2} t, \lambda^{-1} x, \lambda \xi) \xrightarrow{\lambda \to \infty} r^{m-p} b_0(\vartheta, \xi),$$

where the right-hand side is regarded as a function of $(t,x,\xi) \in \Upsilon \times \dot{\mathbb{R}}^d$ *with* $b_0 \in \mathscr{C}^\infty(\Theta, S^{(m-1)}(\dot{\mathbb{R}}^d))$.

Proof. Use the asymptotic expansion with $K = 1$ to obtain

$$|a(t,x,\xi) - r^{m-p} b_0(\vartheta,\xi)| \lesssim \sigma^{m-p+1} |\xi|^m.$$

We are now using the scaling $\kappa_\lambda \colon \Upsilon \times \dot{\mathbb{R}}^d \longrightarrow \Upsilon \times \dot{\mathbb{R}}^d$ via

$$k_\lambda(t,x,\xi) = (\lambda^{-2} t, \lambda^{-1} x, \lambda \xi)$$

for $\lambda > 0$. Note that on $\Theta \times \dot{\mathbb{R}}^d$ the above defined scaling reads as $\kappa_\lambda(\widetilde{t}, \widetilde{x}, \xi) = (\widetilde{t}, \widetilde{x}, \lambda \xi)$. We then have, that

$$\left| a(\lambda^{-2} t, \lambda^{-1} x, \lambda \xi) - \frac{1}{\lambda^{m-p}} r^{m-p} b_0(\vartheta, \lambda \xi) \right| \lesssim \lambda^{p-m-1} |\xi|^m \cdot |\lambda|^m \sigma^{m-p+1},$$

which is nothing but

$$\left| a(\lambda^{-2} t, \lambda^{-1} x, \lambda \xi) - \lambda^{p-m} r^{m-p} b_0(\vartheta, \lambda \xi) \right| \lesssim \lambda^{p-1} |\xi|^m \sigma^{m-p+1}.$$

Since $b_0 \in \mathscr{C}^\infty(\Theta, S^{(m)}(\dot{\mathbb{R}}^d))$, we have the identity $b_0(\vartheta, \lambda\xi) = \lambda^m b_0(\vartheta, \xi)$ and so

$$\left| a(\lambda^{-2}t, \lambda^{-1}x, \lambda\xi) - \lambda^p r^{m-p} b_0(\vartheta, \xi) \right| \lesssim \lambda^{p-1} |\xi|^m \sigma^{m-p+1}.$$

Multiplying by λ^{-p} brings us to

$$\left| \lambda^{-p} a(\lambda^{-2}t, \lambda^{-1}x, \lambda\xi) - r^{m-p} b_0(\vartheta, \xi) \right| \lesssim \frac{|\xi|^m \sigma^{m-p+1}}{\lambda} \longrightarrow 0$$

as $\lambda \longrightarrow \infty$ for all $(t, x, \xi) \in \Upsilon \times \dot{\mathbb{R}}^d$. Hence,

$$\lim_{\lambda \to \infty} \frac{1}{\lambda^p} a(\lambda^{-2}t, \lambda^{-1}x, \lambda\xi) = r^{m-p} b_0(\vartheta, \xi).$$

□

Lemma 3.17. *If $a \in \Sigma^{(m),p}$, then $\chi^+ a \in \Sigma^{m,p}$. In particular, multiplying by χ^+ gives a quantization $\Sigma^{(m),p} \longrightarrow \Sigma^{m,p}$.*

Proof. If $a \in \Sigma^{(m),p}$, we also have the symbol estimate

$$|\partial_t^j \partial_x^\alpha \partial_\xi^\beta a(t, x, \xi)| \lesssim |\xi|^{p+2j+|\alpha|-|\beta|} (\sigma|\xi|)^{m-p-2j-|\alpha|}.$$

Because of $|\xi| \gtrsim \sigma|\xi| \gtrsim 1$, we obtain $\sigma|\xi| \simeq \langle \sigma\xi \rangle$ and $|\xi| \simeq \langle \xi \rangle$, and so $\chi^+ a \in \Sigma^{m,p}$. □

In view of this, we now introduce those symbols of which we can derive principal symbols.

Definition 3.18. For $(m, p) \in \mathbb{R}^2$, the class $\widetilde{\Sigma}^{m,p}$ consists of all $a \in \Sigma^{m,p}$, that can be written in the form

$$a(t, x, \xi) = \chi^+(t, x, \xi)(a_0(t, x, \xi) + a_1(t, x, \xi)) + a_r(t, x, \xi),$$

where $a_0 \in \Sigma^{(m),p}$, $a_1 \in \Sigma^{(m-1),p}$ and $a_r \in \Sigma^{m-1,p-1,\dagger} + \Sigma^{m-2,p,\dagger}$.

Remark 3.19. We have $S_{\mathrm{cl}}([0, T] \times \mathbb{R}^{2d}) \subseteq \widetilde{\Sigma}^{m,m}$.

The next goal is to show that the corresponding class of operators $\mathrm{Op}(\widetilde{\Sigma}^{m,p})$ is closed under taking adjont. The main observation is that the remainder terms are already okay, since they lie in an appropriate class. Let now $a \in \Sigma^{(m),p}$, and so $(\chi^+ a)(t, x, D_x) \in \mathrm{Op}(\Sigma^{m,p})$. We are interested in the question, if the symbol $b(t, x, \xi)$ of the adjoint operator $(\chi^+ a)(t, x, D_x)^\dagger$ is again of the form as in $\widetilde{\Sigma}^{m,p}$. Basically, since for $t > 0$ this theory fits into the standard theory, the most difficulties arise when $t = 0$. Fortunately, then symbols in $\Sigma^{(m),p}$ can be understood as a conormal distribution. Using the asymptotic expansion, we are able to control the arising remainder terms.

Remark 3.20. If E and F are locally convex topological spaces then $E \otimes F$ denotes the algebraic tensor product. The topologies of locally convex topological vector spaces E and F are given by families of seminorms. For each choice of seminorm on E and on F we can define the corresponding family of cross norms on the algebraic tensor product $E \otimes F$, and by choosing one cross norm from each family we get some cross norms on $E \otimes F$, defining a topology. There are in general an enormous number of ways to do this. The two most important ways are to take all the projective cross norms $\otimes_\pi$, or all the injective cross norms $\otimes_\varepsilon$. The completions of the resulting topologies on $E \otimes F$ are called the projective and injective tensor products, and denoted by $E\hat{\otimes}_\pi F$ and $E\hat{\otimes}_\varepsilon F$. There is a natural map from $E\hat{\otimes}_\pi F$ to $E\hat{\otimes}_\varepsilon F$. If one if the factors E and F is nuclear it turns out that $E\hat{\otimes}_\pi F \cong E\hat{\otimes}_\varepsilon F$. For further details on tensor topologies and nuclearity we refer to **[Gro]**.

In the theory of pseudodifferential operators it turns out that the class $S^m(\mathbb{R}^d)$ of symbols of order m is not nuclear, whereas both $S_{\text{cl}}(\mathbb{R}^d)$ and $S^{(m)}(\dot{\mathbb{R}}^d)$ are nuclear Fréchet spaces. Since there is a 1-to-1-correspondence between symbols $a \in \widetilde{\Sigma}^{m,p}$ and functions in the space $\mathscr{C}([0,T], S_{\text{cl}}^{-d-m+p}(\mathbb{R}^d_\eta))\hat{\otimes} S^{(m)}(\dot{\mathbb{R}}^d_\xi)$, we also work in a nuclea Fréchet space.

We will first prove the following lemma:

Lemma 3.21. *Let $m \in \mathbb{R}$ and $a(t,x,\xi) \in \mathscr{C}([0,T], S_{cl}^{-d-m+p}(\mathbb{R}^d_\eta))\hat{\otimes} S^{(m)}(\dot{\mathbb{R}}^d_\xi)$. Then*

$$|\mathscr{F}^{-1}_{\eta\to x} a(t,\eta,\eta+\xi)| \lesssim \langle\sigma\xi\rangle^{m-p}\langle\xi\rangle^p.$$

Proof. Fix $t \in [0,T]$. For simplicity we write $a(t,\eta,\eta+\xi) = c(t,\eta)|\xi+\eta|^m$ with $c(t,\cdot) \in S_{\text{cl}}^{-d-m+p}$ for all $t \in [0,T]$. The general case follows with similar arguments.

Let first $m \leq 0$. Then

$$\mathscr{F}^{-1}_{\eta\to x}\{c(t,\eta)|\xi+\eta|^m\} = \int_{\mathbb{R}^d} e^{ix\eta} c(t,\eta)|\xi+\eta|^m d\eta.$$

We now have the identity

$$(1+t|\xi|^2 - |\xi|^2\Delta_\eta)^{\frac{p-m}{2}} e^{ix\eta} = (1+t|\xi| + |\xi|^2|x|^2)^{\frac{p-m}{2}} e^{ix\eta},$$

which is

$$\langle\sigma\xi\rangle^{m-p}(1+t|\xi|^2 - |\xi|^2\Delta_\eta)^{\frac{p-m}{2}} e^{ix\eta} = e^{ix\eta}.$$

Note that the operator $(1+t|\xi|^2-|\xi|^2\Delta_\eta)^{\frac{p-m}{2}}$ is self-adjoint. Using this identity and integrate the oscillatory integral by part, we arrive at

$$\begin{aligned}&\int_{\mathbb{R}^d} e^{ix\eta}c(t,\eta)|\xi+\eta|^m d\eta\\&=\langle\sigma\xi\rangle^{m-p}\int_{\mathbb{R}^d} e^{ix\eta}(1+t|\xi|^2-|\xi|^2\Delta_\eta)^{\frac{p-m}{2}}(c(t,\eta)|\xi+\eta|^m)\,d\eta\\&=\langle\sigma\xi\rangle^{m-p}\langle\xi\rangle^{p-m}\langle\xi\rangle^m\int_{\mathbb{R}^d} e^{ix\eta}\left(\frac{1}{\langle\xi\rangle^2}+\frac{|\xi|^2(t-\Delta_\eta)}{\langle\xi\rangle^2}\right)^{\frac{p-m}{2}}\left(c(t,\eta)\left|\frac{\xi}{\langle\xi\rangle}+\frac{\eta}{\langle\xi\rangle}\right|^m\right)d\eta.\end{aligned}$$

The operator, including the variable ξ, has at most order zero, so it will not worsen the estimate. Moreover,

$$\left(\frac{1}{\langle\xi\rangle^2}+\frac{|\xi|^2(t-\Delta_\eta)}{\langle\xi\rangle^2}\right)^{\frac{p-m}{2}}\left(c(t,\eta)\left|\frac{\xi}{\langle\xi\rangle}+\frac{\eta}{\langle\xi\rangle}\right|^m\right)\in S_{\mathrm{cl}}^{-d+m}\subset S_{\mathrm{cl}}^{-d},$$

uniformly for every $t\in[0,T]$ and so

$$\left|\int_{\mathbb{R}^d} e^{ix\eta}\left(\frac{1}{\langle\xi\rangle^2}+\frac{|\xi|^2(t-\Delta_\eta)}{\langle\xi\rangle^2}\right)^{\frac{p-m}{2}}\left(c(t,\eta)\left|\frac{\xi}{\langle\xi\rangle}+\frac{\eta}{\langle\xi\rangle}\right|^m\right)d\eta\right|\lesssim 1.$$

This means, that

$$|\mathscr{F}^{-1}_{\eta\to x}a(t,\eta,\eta+\xi)|\lesssim\langle\sigma\xi\rangle^{m-p}\langle\xi\rangle^p.$$

Let now $m\geq 0$. Then

$$|\xi+\eta|^m\lesssim\sum_{k=0}^m|\xi|^{m-k}|\eta|^k,$$

so we have to consider

$$\sum_{k=0}^m\int_{\mathbb{R}^d} e^{ix\eta}c(t,\eta)|\xi|^{m-k}|\eta|^k d\eta.$$

Now fix a $k\in\{0,1,\dots,m\}$. Then, with the same computations as above, we first have $|\eta|^k c(t,\eta)\in S_{\mathrm{cl}}^{-d-(m-k)+p}$, and also

$$\begin{aligned}\left|\int_{\mathbb{R}^d} e^{ix\eta}c(t,\eta)|\xi|^{m-k}|\eta|^k d\eta\right|&\lesssim\langle\sigma\xi\rangle^{m-k-p}\langle\xi\rangle^p\\&\lesssim\langle\sigma\xi\rangle^{m-p}\langle\xi\rangle^p.\end{aligned}$$

So, for every summand we first derive a stronger estimate, which can be bounded from above by $\langle\sigma\xi\rangle^{m-p}\langle\xi\rangle^p$ for all k. This completes the proof. $\square$

This enables us to state the following theorem:

Theorem 3.22. *Let $a\in\Sigma^{(m),p}$ and let $b(t,x,\xi)$ the full symbol of the adjoint operator $(\chi^+a)(t,x,D_x)^\dagger$. Then $b\in\widetilde{\Sigma}^{m,p}$.*

Proof. If we consider t as a parameter, then by the usual techniques (see for instance [**Hor3**]) the Fourier transform of b fulfills the identity

$$\hat{b}(t, \xi - \eta, \eta) = (\widehat{\chi^+ a})^*(t, \eta - \xi, \xi),$$

where the hat denotes the Fourier transform with respect to x and covariable η. Noting, that χ^+ is a scalar and real function, we get

$$\hat{b}(t, \xi - \eta, \eta) = (\chi^+ a^*)^\wedge(t, \eta - \xi, \xi).$$

Then, by change of variables, we obtain

$$\hat{b}(t, \eta, \xi) = (\chi^+ a^*)^\wedge(t, -\eta, \eta + \xi),$$

and an application of Taylor's expansion brings us to

$$\begin{aligned}\hat{b}(t, \eta, \xi) &= (\chi^+ a^*)^\wedge(t, -\eta, \xi) + \partial_\xi(\chi^+ a^*)^\wedge(t, -\eta, \xi)\eta \\ &\qquad + 2\sum_{|\alpha|=2} \eta^\alpha \int_0^1 (1-\theta)\partial_\xi^\alpha(\chi^+ a^*)^\wedge(t, -\eta, \xi + \theta\eta)\, d\theta \\ &= (I_1 + I_2 + I_3)(t, \eta, \xi).\end{aligned}$$

We now transform back the expression I_1 und I_2. Then we get

$$\mathscr{F}^{-1}_{\eta\to x}(I_1) = \int_{\mathbb{R}^d} (\chi^+ a^*)^\wedge(t, -\eta, \xi) e^{ix\eta}\, d\eta = (\chi^+ a^*)(t, x, \xi)$$

and in a similar fashion also

$$\mathscr{F}^{-1}_{\eta\to x}(I_2) = i\partial_x\partial_\xi(\chi^+ a^*)(t, x, \xi) = i\chi^+\partial_x\partial_\xi a^*(t, x, \xi) \mod \Sigma^{-\infty,p}.$$

For the remainder term

$$r(t, \eta, \xi) = 2\sum_{|\alpha|=2} \eta^\alpha \int_0^1 (1-\theta)\partial_\xi^\alpha(\chi^+ a^*)^\wedge(t, -\eta, \xi + \theta\eta)\, d\theta$$

we will show $\mathscr{F}^{-1}_{\eta\to x}(r(t, \eta, \xi)) \in \Sigma^{m-2,p,\dagger}$. By Fubini's theorem, it suffices to prove the desired estimate, uniformly in $\theta \in [0, 1]$, for any term

$$\eta^\alpha \partial_\xi^\alpha(\chi^+ a^*)^\wedge(t, -\eta, \xi + \theta\eta), \qquad |\alpha| = 2.$$

Since $a^* \in \Sigma^{(m),p}$, it has an expansion of the form

$$a^* \sim \sum r^{m-p+k} b_0\,(\vartheta, \xi)$$

with $b_0 \in \mathscr{C}^\infty(\Theta \times S^{(m)}(\dot{\mathbb{R}}^d))$. Hence,

$$\mathscr{F}^{-1}_{\eta\to x}(a^*(t,\eta,\xi)) \in \mathscr{C}^\infty((0,T],\mathscr{S}(\mathbb{R}^{2d}_{\eta,\xi})) \subset \mathscr{C}([0,T],S_{\mathrm{cl}}^{-d-m+p}(\mathbb{R}^d_\eta))\hat{\otimes}S^{(m)}(\dot{\mathbb{R}}^d_\xi).$$

Since convolution does not have an effect, we have

$$\eta^\alpha\partial_\xi^\alpha(\chi^+a^*)^\wedge \in \mathscr{C}^\infty((0,T],\mathscr{S}(\mathbb{R}^{2d}_{\eta,\xi})) \subset \mathscr{C}([0,T],S_{\mathrm{cl}}^{-d-(m-2)+p}(\mathbb{R}^d_\eta))\hat{\otimes}S^{(m-2)}(\dot{\mathbb{R}}^d_\xi),$$

as $|\alpha|=2$. An application of Lemma 3.21 leads to the desired result. Note that the estimates in that lemma are still true with that parameter θ. Integration over θ completes the proof. □

We are now able to state the following important result.

Theorem 3.23. *The class* $\mathrm{Op}(\widetilde{\Sigma}^{m,p})$ *is closed under taking adjoints.*

Proof. It is

$$a(t,x,D_x)^\dagger = (\chi^+a_0)(t,x,D_x)^\dagger + (\chi^+a_1)(t,x,D_x)^\dagger + a_r(t,x,D_x)^\dagger.$$

The first two terms belong to $\mathrm{Op}(\widetilde{\Sigma}^{m,p})$ by Theorem 3.22, the third by definition of the class $\widetilde{\Sigma}^{m,p}$. □

We immediately obtain

Corollary 3.24. *It holds* $\mathrm{Op}(\widetilde{\Sigma}^{m,p}) \subseteq \mathrm{Op}(\Sigma^{m,p,\dagger})$.

When a class of pseudodifferential operators is closed under taking adjoint, one similarly gets, that also compositions are welldefined. We next show the following theorem, by using the same techniques as in Lemma 3.21.

Theorem 3.25. *Let* $a \in \Sigma^{(m),p}$ *and* $b \in \Sigma^{(m'),p'}$ *and let* $c(t,x,\xi)$ *the full symbol of the composition operator* $((\chi^+a)\#(\chi^+b))(t,x,D_x)$*. Then* $c \in \widetilde{\Sigma}^{m+m',p+p'}$.

Proof. Let $a \in \Sigma^{(m),p}$, $b \in \Sigma^{(m'),p'}$. Then χ^+a and χ^+b belong to $\Sigma^{m,p}$ and $\Sigma^{m',p'}$, respectively. By standard computation, i.e. applying the Fourier transform (see for example [**Hor3**]), the full symbol $c(t,x,\xi)$ of

$$c(t,x,D_x) = (\chi^+a)(t,x,D_x)\#(\chi^+b)(t,x,D_x)$$

is given by

$$c(t,x,\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{-iy\mu}(\chi^+a)(t,x,\xi+\mu)(\chi^+b)(t,x+y,\xi)\,dy\,d\mu.$$

An Taylor expansion of $(\chi^+a)(t,x,\xi+\mu)(\chi^+b)(t,x+y,\xi)$ at $\mu=0$ yields

$$\begin{aligned}(\chi^+a)(t,x,\xi+\mu)(\chi^+b)(t,x+y,\xi) = (\chi^+a)(t,x,\xi)(\chi^+b)(t,x+y,\xi)\\ +\partial_\xi(\chi^+a)(t,x,\xi+\mu)(\chi^+b)(t,x+y,\xi)\mu\\ +r(t,x,\xi,y,\mu),\end{aligned}$$

where

$$r(t,x,\xi,y,\mu) = 2\sum_{|\alpha|=2}\frac{1}{\alpha!}\int_0^1(1-\theta)(\partial_\xi^\alpha(\chi^+a))(t,x,\xi+\mu)(\chi^+b)(t,x+y,\xi)\mu^\alpha\,d\theta.$$

Since, as iterated integral,

$$\begin{aligned}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{-iy\mu}\partial_\xi^\alpha(\chi^+a)(t,x,\xi+\mu)(\chi^+b)(t,x+y,\xi)\mu\,dy\,d\mu\\ =\partial_\xi^\alpha(\chi^+a)(t,x,\xi)D_x^\alpha(\chi^+b)(t,x,\xi),\end{aligned}$$

for $|\alpha|\le 1$, it suffices to show, uniformly with respect to $\theta\in[0,1]$, that for every term

$$\hat{r}(t,x,\xi)=\frac{1}{(2\pi)^d}\left(\partial_\xi^\alpha(\chi^+a)\right)(t,x,\xi+\theta\mu)(\chi^+b)(t,x+y,\xi)\mu^\alpha\,dyd\mu$$

with $|\alpha|=2$ we have

$$|r(t,x,\xi)|\le\langle\sigma\xi\rangle^{m+m'-2-p-p'}\langle\xi\rangle^{m+m'-2}.$$

But this follows immediately with a similar analysis as in Lemma 3.21. Integration over θ gives the result. Thus, c can be written as

$$c(t,x,\xi)=\chi^+(ab-i\partial_\xi a\partial_x b)+c_r,$$

with $ab\in\Sigma^{(m+m'),p+p'}$ and $\partial_\xi a\partial_x b\in\Sigma^{(m+m'-1),p+p'}$ □

The next important consequence is:

Theorem 3.26. *The class* $\mathrm{Op}(\widetilde{\Sigma}^{m,p})$ *is closed under taking compositions.*

Proof. For $A\in\mathrm{Op}(\widetilde{\Sigma}^{m,p})$ and $B\in\mathrm{Op}(\widetilde{\Sigma}^{m',p'})$ we write

$$A(t,x,D_x)=(\chi^+a_0)(t,x,D_x)+(\chi^+a_1)(t,x,D_x)+a_r(t,x,D_x)$$

and

$$B(t,x,D_x)=(\chi^+b_0)(t,x,D_x)+(\chi^+b_1)(t,x,D_x)+b_r(t,x,D_x),$$

respectively. Then

$$
\begin{aligned}
(A\#B)(t,x,D_x) = \sum_{j,k=0}^{1} &((\chi^+ a_j)\#(\chi^+ b_k))(t,x,D_x) \\
&+ \sum_{j=0}^{1} \Big((\chi^+ a_j)\#b_r + a_r\#(\chi^+ b_j)\Big)(t,x,D_x) \\
&+ (a_r\#b_r)(t,x,D_x).
\end{aligned}
$$

By Theorem 3.25, the first sum belongs to $\mathrm{Op}(\widetilde{\Sigma}^{m+m',p+p'})$. Since $a_r, b_r \in \Sigma^{m-2,p,\dagger} + \Sigma^{m-1,p-1,\dagger}$, adjoints are well-defined, and so compositions. Hence, the full symbol of remainder term belongs to $\Sigma^{m+m'-2,p+p',\dagger} + \Sigma^{m+m'-1,p+p'-1,\dagger}$. □

With operators $A \in \mathrm{Op}(\widetilde{\Sigma}^{m,p})$ we now associate the following principal symbols:

$$
\begin{aligned}
\sigma_\Psi^m(A) &:= a_0 \\
\sigma_{\Psi,d}^{m,p}(A) &:= \lim_{\lambda\to\infty} \lambda^{-p} a_0(\lambda^{-2}t, \lambda^{-1}x, \lambda\xi) \\
\sigma_{\Psi,d}^{m-1,p}(A) &:= \lim_{\lambda\to\infty} \lambda^{-p} a_1(\lambda^{-2}t, \lambda^{-1}x, \lambda\xi).
\end{aligned}
$$

Let us discuss some examples:

Example 3.27.

(i) For $\Lambda^{m,p} = \lambda^{m,p}(t,x,D_x)$, where $\lambda^{m,p}(t,x,D_x) = \langle\sigma\xi\rangle_K^{m-p}\langle\xi\rangle_K^p$, K sufficiently large, we have

$$
\langle\sigma\xi\rangle^{m-p}\langle\xi\rangle^p = \sigma^{m-p}|\xi|^m + O\left(|\xi|^{m-2}\right) \qquad \text{as } |\xi| \longrightarrow \infty.
$$

Thus

$$
\sigma_\Psi^m(\Lambda^{m,p}) = \sigma^{m-p}|\xi|^m, \ \ \sigma_{\Psi,d}^{m,p}(\Lambda^{m,p}) = r^{m-p}|\xi|^m, \ \ \sigma_{\Psi,d}^{m-1,p}(\Lambda^{m,p}) = 0.
$$

(ii) If $a_0 \in S^{(m)}$ is the principal symbol of $A \in \mathrm{Op}(S^m_{\mathrm{cl}})$, then

$$
\sigma_\Psi^m(A) = a_0(t,x,\xi), \ \ \sigma_{\Psi,d}^{m,m}(A) = a_0(0,0,\xi), \ \ \sigma_{\Psi,d}^{m-1,m}(A) = 0.
$$

The introduction of the principal symbols $\sigma_\Psi^m(a)$ and $\sigma_{\Psi,d}^{m-1,p}(a)$ is justified by the next lemma:

Lemma 3.28.

(i) The symbols $\sigma_\Psi^m(A)$ and $\sigma_{\Psi,d}^{m-1,p}(A)$ are well-defined.

(ii) The short sequence

$$0 \longrightarrow \mathrm{Op}(\Sigma^{m-2,p,\dagger}) + \mathrm{Op}(\Sigma^{m-1,p-1,\dagger}) \longrightarrow \mathrm{Op}(\widetilde{\Sigma}^{m,p})$$
$$\xrightarrow{(\sigma_\Psi^m,\sigma_{\Psi,d}^{m-1,p})} \Sigma^{(m),p} \times \Sigma^{(m-1,p)} \longrightarrow 0$$

is exact, where

$$\Sigma^{(m,p)} := r^{m-p}\mathscr{C}^\infty(\Theta, S^{(m)}(\dot{\mathbb{R}}^d))$$

is the space of (m,p)-bihomogeneous symbols.

Proof. We work in the corresponding symbol classes.

Statement (i) is obvious by definition of the class $\widetilde{\Sigma}^{m,p}$. Given $a \in \Sigma^{m,p}$, the principal symbol is uniquely determined by the structure of a. The other symbols are welldefined by the uniqueness of the limit.

Let us now prove (ii). The surjectivity of the map $(\sigma_\Psi^m, \sigma_{\Psi,d}^{m-1,p})$ is quite easy. Given any $a_0 \in \Sigma^{(m),p}$ and $a_1 \in \Sigma^{(m,p)} \subseteq \Sigma^{(m),p}$, set $a = \chi^+(a_0 + a_1)$ with $a_r \equiv 0$. Then $a \in \widetilde{\Sigma}^{m,p}$. It is left to prove the exactness in the middle of the sequence. Therefore, we are going to show

$$a \in \Sigma^{m-2,p,\dagger} + \Sigma^{m-1,p-1,\dagger} \iff \sigma_\Psi^m(a) = 0,\;\; \sigma_{\Psi,d}^{m-1,p}(a) = 0.$$

Let first $\sigma_\Psi^m(a) = 0, \sigma_{\Psi,d}^{m-1,p}(a) = 0$. Then $a_1 \in \Sigma^{(m-1),p-1}$ and so $\chi^+ a_1 \in \Sigma^{(m-1),p-1,\dagger}$. Thus, $a = \chi^+ a + a_r \in \Sigma^{m-1,p-1,\dagger}$.

Let now $a_0 \neq 0$. Then $|a| \geq C^{-1}\langle\sigma\xi\rangle^{m-p}\langle\xi\rangle^p$ for $\sigma\langle\xi\rangle \geq C$ in some conic set and $C > 0$ sufficiently large. But then $a \notin \Sigma^{m-2,p,\dagger} + \Sigma^{m-1,p-1,\dagger}$. If $a_0 = 0$ and $\sigma_{\Psi,d}^{m,p-1}(a) \neq 0$, then $a_1 \in \Sigma^{(m-1),p}$. Hence, $a \in \Sigma^{m-1,p,\dagger}$ and so $a \notin \Sigma^{m-2,p,\dagger} + \Sigma^{m-1,p-1,\dagger}$.

□

Remark 3.29. We briefly summarize what vanishing of the single symbolic components for $a \in \widetilde{\Sigma}^{m,p}$ means:

$$\begin{aligned} \sigma_\Psi^m(a) = 0,\;\; \sigma_{\Psi,d}^{m-1,p}(a) = 0 &\iff a \in \Sigma^{m-2,p,\dagger} + \Sigma^{m-1,p-1,\dagger} \\ \sigma_\Psi^m(a) = 0 &\iff a \in \Sigma^{m-1,p,\dagger} \\ \sigma_{\Psi,d}^{m,p}(a) = 0 &\iff a \in \Sigma^{m-1,p,\dagger} + \Sigma^{m,p-1,\dagger}. \end{aligned}$$

For the principal symbols we have the following rules:

Proposition 3.30.

(i) If $A \in \operatorname{Op}(\widetilde{\Sigma}^{m,p})$ *then*

$$\sigma_\Psi^m(A^\dagger) = \sigma_\Psi^m(A)^*, \qquad \sigma_{\Psi,d}^{m,p}(A^\dagger) = \sigma_{\Psi,d}^{m,p}(A)^*$$

and

$$\sigma_{\Psi,d}^{m-1,p}(A^\dagger) = \sigma_{\Psi,d}^{m-1,p}(A)^* - i\partial_\xi \partial_x \sigma_{\Psi,d}^{m,p}(A)^*.$$

(ii) If $A \in \operatorname{Op}(\widetilde{\Sigma}^{m,p})$ *and* $B \in \operatorname{Op}(\widetilde{\Sigma}^{m',p'})$*, then*

$$\begin{aligned}
\sigma_\Psi^{m+m'}(A\#B) &= \sigma_\Psi^m(A)\sigma_\Psi^{m'}(B)\\
\sigma_{\Psi,d}^{m+m',p+p'}(A\#B) &= \sigma_{\Psi,d}^{m,p}(A)\sigma_{\Psi,d}^{m',p'}(B)\\
\sigma_{\Psi,d}^{m+m'-1,p+p'}(A\#B) &= \sigma_{\Psi,d}^{m,p}(A)\sigma_{\Psi,d}^{m'-1,p'}(B) + \sigma_{\Psi,d}^{m-1,p}(A)\sigma_{\Psi,d}^{m',p'}(B)\\
&\quad - i\partial_\xi \sigma_{\Psi,d}^{m,p}(A)\partial_x \sigma_{\Psi,d}^{m',p'}(B).
\end{aligned}$$

(iii) If $A \in \operatorname{Op}(\widetilde{\Sigma}^{m,p})$ *is elliptic in the sense that*

$$|\det \sigma_\Psi^m(A)| \geq c(\sigma^{m-p}|\xi|^m)^N$$

for some $c > 0$*, then* $A^{-1} \in \operatorname{Op}(\widetilde{\Sigma}^{-m,-p})$ *and*

$$\sigma_\Psi^{-m}(A^{-1}) = \sigma_\Psi^m(A)^{-1}, \qquad \sigma_{\Psi,d}^{-m,-p}(A^{-1}) = \sigma_{\Psi,d}^{m,p}(A)^{-1}$$

and

$$\begin{aligned}
\sigma_{\Psi,d}^{-m-1,-p}(A^{-1}) = &-i\sigma_{\Psi,d}^{m,p}(A)^{-1}\partial_\xi \sigma_{\Psi,d}^{m,p}(A)\sigma_{\Psi,d}^{m,p}(A)^{-1}\sigma_{\Psi,d}^{m,p}(A)\partial_x \sigma_{\Psi,d}^{m,p}(A)^{-1}\\
&- \sigma_{\Psi,d}^{m,p}(A)^{-1}\sigma_{\Psi,d}^{m-1,p}(A)\sigma_{\Psi,d}^{m,p}(A)^{-1}
\end{aligned}$$

(iv) If $A \in \operatorname{Op}(\widetilde{\Sigma}^{m,p})$*, then* $[D_t, A] = (D_t A) \in \operatorname{Op}(\widetilde{\Sigma}^{m,p+2})$ *and*

$$\sigma_\Psi^m(D_t A) = D_t \sigma_\Psi^m(A), \qquad \sigma_{\Psi,d}^{m,p+2}(D_t A) = D_t \sigma_{\Psi,d}^{m,p}(A)$$

as well as

$$\sigma_{\Psi,d}^{m-1,p+2}(D_t A) = D_t \sigma_{\Psi,d}^{m-1,p}(A).$$

Proof. The formulas for (i) and (ii) follow immediately from Theorem 3.23 and Theorem 3.26.

If A^{-1} is the right-inverse to A, then $A\#A^{-1} = I$ and so by (ii)

$$E = \sigma_\Psi^0(I) = \sigma_\Psi^0(A\#A^{-1}) = \sigma_\Psi^m(A)\sigma_\Psi^{-m}(A^{-1}).$$

Hence, $\sigma_\Psi^{-m}(A^{-1}) = \sigma_\Psi^m(A)^{-1}$. Using this we also obtain the formula $\sigma_{\Psi,d}^{-m,-p}(A^{-1}) = \sigma_{\Psi,d}^{m,p}(A)^{-1}$. Since $\sigma_{\Psi,d}^{-1,0}(I) = 0$, we get

$$0 = \sigma_{\Psi,d}^{m,p}(A)\sigma_{\Psi,d}^{-m-1,-p}(A^{-1}) + \sigma_{\Psi,d}^{m-1,p}(A)\sigma_{\Psi,d}^{-m,-p}(A^{-1}) - \partial_\xi\sigma_{\Psi,d}^{m,p}(A)\partial_x\sigma_{\Psi,d}^{-m,-p}(A^{-1}).$$

Note that by chain rule, we have

$$\partial_x\sigma_{\Psi,d}^{-m,-p}(A^{-1}) = -\sigma_{\Psi,d}^{m,p}(A)^{-1}\partial_x\sigma_{\Psi,d}^{m,p}(A)\sigma_{\Psi,d}^{m,p}(A)^{-1}.$$

Using this, isolating to $\sigma_{\Psi,d}^{-m-1,-p}(A^{-1})$ and applying the previous formulas, gets us the desired result.

If $A = \mathrm{Op}(a)$ for an $a \in \widetilde{\Sigma}^{m,p}$, then $[D_t, A] = \mathrm{Op}(D_t a)$. By using the structure of a this yields $\sigma_\Psi^m(D_t a) = D_t\sigma_\Psi^m(a)$. Moreover,

$$\begin{aligned}
\sigma_{\Psi,d}^{m,p+2}([D_t, A]) &= \lim_{\lambda\to\infty} \frac{1}{\lambda^{p+2}}(D_t a_0)(\lambda^{-2}t, \lambda^{-1}x, \lambda\xi) \\
&= \lim_{\lambda\to\infty} \frac{1}{\lambda^p} \cdot \frac{1}{\lambda^2}(D_t a_0)(\lambda^{-2}t, \lambda^{-1}x, \lambda\xi) \\
&= \lim_{\lambda\to\infty} \frac{1}{\lambda^p} D_t a_0(\lambda^{-2}t, \lambda^{-1}x, \lambda\xi) \\
&= D_t\sigma_{\Psi,d}^{m,p}(A).
\end{aligned}$$

The same argument holds for the other symbol. □

Remark 3.31. We shorten the notation in the following way: For an operator $A \in \mathrm{Op}(\widetilde{\Sigma}^{m,p})$ we write

$$\sigma_\Psi^m(A) = A_0, \qquad \sigma_{\Psi,d}^{m,p}(A) = A_{00} \quad \text{and} \quad \sigma_{\Psi,d}^{m-1,p}(A) = A_{01}$$

and $D_x = -i\partial_x$. Then, for compositions and inverses, we obtain

$$(AB)_{01} = A_{00}B_{01} + A_{01}B_{00} - i(A_{00})_\xi(B_{00})_x$$

and

$$(A_{01}^{-1}) = -(A_{00})^{-1}A_{01}(A_{00})^{-1} - i(A_{00})^{-1}(A_{00})_\xi(A_{00})^{-1}(A_{00})_x(A_{00})^{-1}.$$

Consequently, for three operators A, B and C, we get

$$\begin{aligned}
(ABC)_{01} = {} & A_{00}B_{00}C_{01} + A_{00}B_{01}C_{00} + A_{01}B_{00}C_{00} \\
& - i(A_{00})_\xi(B_{00})_x C_{00} - i(A_{00})_\xi B_{00}(C_{00})_x - iA_{00}(B_{00})_\xi(C_{00})_x.
\end{aligned}$$

3.4 Function spaces

In this section we provide the associated function spaces, which come from the original symbol estimate. Moreover, we are going to prove the corresponding mapping properties.

For $K \geq 1$ define $\langle \xi \rangle_K := \sqrt{K^2 + |\xi|^2}$, and set $\lambda_K^{m,p}(t,x,\xi) := \langle \xi \rangle_K^p \langle \sigma\xi \rangle_K^{m-p}$. Then it is clear, that $\lambda_K^{m,p} \in \widetilde{\Sigma}^{m,p}$ for every $K \geq 1$. Similar to the estimates in $\Sigma^{m,p}$ one can derive, that

$$|\partial_t^j \partial_x^\alpha \partial_\xi^\beta \lambda_K^{m,p}| \lesssim \langle \xi \rangle_K^{p+2j-|\beta|+|\alpha|} \langle \sigma\xi \rangle_K^{m-p-|\alpha|-2j}. \tag{3.2}$$

Denote $\Lambda_K^{m,p} = \mathrm{Op}(\lambda_K^{m,p})$. We are going to prove the following proposition:

Proposition 3.32. *Given $(m,p) \in \mathbb{Z} \times \mathbb{R}$ there is a $K_0 \geq 1$, such that the operator $\Lambda_K^{m,p} \in \mathrm{Op}(\widetilde{\Sigma}^{m,p})$ is invertible for all $K \geq K_0$ with $(\Lambda_K^{m,p}) \sim \Lambda_K^{-m,-p}$ modulo smoothing terms.*

This is an application of the parameter-dependent pseudodifferential calculus, see for instance [**Shu**]. The idea is to consider operators, depending on a parameter K. Usually, when constructing parametrices, one gets remainder terms of lower order. In this case, the parameter K can be taken sufficiently large such that the norm of that remainder term is strictly less than one. Hence, the operator becomes invertible.

Proof. The symbol $\lambda_K^{m,p}$ belongs to $\widetilde{\Sigma}^{m,p}$. Similarly $\lambda_K^{-m,-p}$ is in $\widetilde{\Sigma}^{-m,-p}$. Define

$$R_K := \lambda_K^{m,p} \# \lambda_K^{-m,-p} - \lambda_K^{m,p} \lambda_K^{-m,-p} = \lambda_K^{m,p} \# \lambda_K^{-m,-p} - 1.$$

For any $\ell \in \mathbb{N}_0$ let $(j,\alpha,\beta) \in \mathbb{N}_0^{1+2d}$ with $j + |\alpha| + |\beta| \leq \ell$. We want to show

$$|R_K|_{0,0;\ell} \leq \frac{C_\ell}{K},$$

where

$$|R_K|_{0,0;\ell} = \sup_{\substack{(t,x,\xi) \in [0,T] \times \mathbb{R}^{2d} \\ j+|\alpha|+|\beta| \leq \ell}} \langle \xi \rangle^{2j+|\beta|-|\alpha|} \langle \sigma\xi \rangle^{|\alpha|+2j} |\partial_t^j \partial_x^\alpha \partial_\xi^\beta R_K|.$$

We will prove $R_K \longrightarrow 0$ in $\Sigma^{0,0}$ for $K \longrightarrow \infty$. By Leibniz rule, it is

$$|\partial_t^j \partial_x^\alpha \partial_\xi^\beta R_K| \leq \sum_{|\gamma| \geq 1} \frac{1}{\gamma!} \sum_{j'+|\alpha'|+|\beta'| \leq \ell} c_{j'\alpha'\beta'} |\partial_t^{j'} \partial_x^{\alpha'} \partial_\xi^{\beta'+\gamma} \lambda_K^{m,p}| |\partial_t^{j-j'} \partial_x^{\alpha-\alpha'+\gamma} \partial_\xi^{\beta-\beta'} \lambda_K^{-m,-p}|.$$

The estimates (3.2), gives that the product in the inner sum can be estimated by the function

$$\langle \xi \rangle_K^{2j-|\beta|+|\alpha|} \langle \sigma\xi \rangle_K^{-|\alpha|-2j-|\gamma|}$$

and so

$$\begin{aligned} |\partial_t^j \partial_x^\alpha \partial_\xi^\beta R_K| &\leq C \langle \xi \rangle_K^{2j-|\beta|+|\alpha|} \langle \sigma\xi \rangle_K^{-|\alpha|-2j} \sum_{|\gamma|\geq 1} \frac{1}{\gamma!} \langle \sigma\xi \rangle^{-|\gamma|} \\ &\leq C_\ell \langle \xi \rangle_K^{2j-|\beta|+|\alpha|} \langle \sigma\xi \rangle_K^{-|\alpha|-2j-1}. \end{aligned}$$

Thus, we obtain

$$|R_K|_{0,0;\ell} \leq C_\ell \sup_{\substack{(t,x,\xi)\in[0,T]\times\mathbb{R}^{2d} \\ j+|\alpha|+|\beta|\leq \ell}} \langle \xi \rangle^{2j+|\beta|-|\alpha|} \langle \sigma\xi \rangle^{|\alpha|+2j} \langle \xi \rangle_K^{2j-|\beta|+|\alpha|} \langle \sigma\xi \rangle_K^{-|\alpha|-2j-1}.$$

Note that $\langle \xi \rangle^{|\beta|} / \langle \xi \rangle_K^{|\beta|} \leq 1$ for all $\beta \in \mathbb{N}_0^d$ and $K \geq 1$, so the previous estimate can be rewritten, by sorting the powers, as

$$|R_K|_{0,0;\ell} \leq C_\ell \sup_{\substack{(t,x,\xi)\in[0,T]\times\mathbb{R}^{2d} \\ j+|\alpha|+|\beta|\leq \ell}} \left(\frac{\langle \sigma\xi \rangle^2 \langle \xi \rangle_K^2}{\langle \sigma\xi \rangle_K^2 \langle \xi \rangle^2} \right)^{\frac{2j+|\alpha|}{2}} \langle \sigma\xi \rangle_K^{-1}.$$

Since

$$\frac{\langle \sigma\xi \rangle^2 \langle \xi \rangle_K^2}{\langle \sigma\xi \rangle_K^2 \langle \xi \rangle^2} \leq 1$$

for all $(t, x, \xi) \in [0, T] \times \mathbb{R}^{2d}$, the estimate reduces to

$$|R_K|_{0,0;\ell} \leq C_\ell \sup_{(t,x,\xi)\in[0,T]\times\mathbb{R}^{2d}} \langle \sigma\xi \rangle_K^{-1} = \frac{C_\ell}{K}.$$

Hence, $R_K \longrightarrow 0$ in $\Sigma^{0,0}$ for $K \longrightarrow \infty$. This implies $R_K(t, x, D_x) \longrightarrow 0$ in $\mathscr{L}(L^2((0,T) \times \mathbb{R}^d)$ for $K \longrightarrow \infty$. Thus $\Lambda_K^{-m,-p} \circ (1 + R_K)^{-1}$ is a right inverse to $\Lambda_K^{m,p}$ for large $K \geq 1$. In a similar way, a left inverse to $\Lambda_K^{m,p}$ is contructed. Moreover, $(\Lambda_K^{m,p})^{-1} = \Lambda_K^{-m,-p}$ modulo smoothing terms, as it is seen from the construction. ☐

We extend the Example 3.27 since we now know that the operators $\Lambda^{m,p}$ are invertible.

Example 3.33. For $\Lambda^{m,p} = \lambda^{m,p}(t, x, D_x)$, where $\lambda^{m,p}(t, x, D_x) = \langle \sigma\xi \rangle_K^{m-p} \langle \xi \rangle_K^p$, K sufficiently large, we have

$$\sigma_\Psi^m(\Lambda^{m,p}) = \sigma^{m-p} |\xi|^m, \quad \sigma_{\Psi,d}^{m,p}(\Lambda^{m,p}) = r^{m-p} |\xi|^m, \quad \sigma_{\Psi,d}^{m-1,p}(\Lambda^{m,p}) = 0.$$

Moreover it holds $(\Lambda^{m,p})^{-1} \in \mathrm{Op}(\widetilde{\Sigma}^{-m,-p})$ and, by using the composition rules,

$$\sigma_{\Psi}^{-m}((\Lambda^{m,p})^{-1}) = \sigma^{p-m}|\xi|^{-m}, \quad \sigma_{\Psi,d}^{-m,-p}((\Lambda^{m,p})^{-1}) = r^{p-m}|\xi|^{-m},$$

and

$$\sigma_{\Psi,d}^{-m-1,-p}((\Lambda^{m,p})^{-1}) = im(p-m)\langle\xi, x\rangle r^{p-m-2}|\xi|^{-m-2}.$$

We now consider those invertible operators and always neglect the parameter K. Let us define the following function spaces:

Definition 3.34. For $s \in \mathbb{N}_0$, $\delta \in \mathbb{R}$ and we define

$$\mathscr{C}H^{s,\delta} = \{u\colon (0,T) \times \mathbb{R}^d \longrightarrow \mathbb{C} \mid \Lambda^{s,\delta}u \in \mathscr{C}([0,T], L^2(\mathbb{R}^d))\}$$

and

$$L^1H^{s,\delta} = \{u\colon (0,T) \times \mathbb{R}^d \longrightarrow \mathbb{C} \mid \Lambda^{s,\delta}u \in L^1((0,T), L^2(\mathbb{R}^d))\},$$

respectively.

We are able to formulate the following mapping property result:

Proposition 3.35. *For $m \in \mathbb{N}_0$ we have*

$$\Lambda^{m,p} \in \mathscr{L}(\mathscr{C}H^{s+m,\delta+p}, \mathscr{C}H^{s,\delta}).$$

for all $(s,\delta) \in \mathbb{N}_0 \times \mathbb{R}$.

Proof. Let $s \in \mathbb{N}_0$. Then we have to show that $\Lambda^{m,p}u \in \mathscr{C}H^{s,\delta}$, which is $\Lambda^{s,\delta}(\Lambda^{m,p}u) \in \mathscr{C}([0,T], L^2(\mathbb{R}^d))$, provided $u \in \mathscr{C}H^{s+m,\delta+p}$. By the composition of two operators we get

$$\Lambda^{s,\delta}\Lambda^{m,p} = \Lambda^{s+m,\delta+p} + R,$$

where $R \in \mathrm{Op}(\widetilde{\Sigma}^{s+m-1,\delta+p})$. By assumption, it holds $\Lambda^{s+m,\delta+p}u \in \mathscr{C}([0,T], L^2(\mathbb{R}^d))$. We now write the remainder term as

$$R = R(\Lambda^{s+m,\delta+p})^{-1}\Lambda^{s+m,\delta+p},$$

since the operators $\Lambda^{s+m,\delta+p}$ are invertible. It turns out that

$$R(\Lambda^{s+m,\delta+p})^{-1} \in \mathrm{Op}(\widetilde{\Sigma}^{-1,0}) \subseteq \mathrm{Op}(\widetilde{\Sigma}^{0,0}).$$

In view of Corollary 3.14, we also have

$$Ru = R(\Lambda^{s+m,\delta+p})^{-1}\Lambda^{s+m,\delta+p}u \in \mathscr{C}([0,T], L^2(\mathbb{R}^d)).$$

This completes the proof. $\square$

Remark 3.36. We get a similar result for $L^1 H^{s,\delta}$.

Let us now come to the main function spaces involving time derivatives.

Definition 3.37. For $s \in \mathbb{N}_0$, $\delta \in \mathbb{R}$ and we define

$$X^{s,\delta} = \{u\colon (0,T) \times \mathbb{R}^d \longrightarrow \mathbb{C} \,|\, \Lambda^{s-j,\delta-2j} D_t^j u \in \mathscr{C}([0,T], L^2(\mathbb{R}^d)),\, 0 \le j \le s\}$$

and

$$Y^{s,\delta} = \{u\colon (0,T) \times \mathbb{R}^d \longrightarrow \mathbb{C} \,|\, \Lambda^{s-j,\delta-2j} D_t^j u \in L^1((0,T), L^2(\mathbb{R}^d)),\, 0 \le j \le s\},$$

respectively. The corresponding norms are given by

$$\|u\|_{X^{s,\delta}} = \sup_{0\le j\le s} \sup_{t\in[0,T]} \|(\langle \sigma D_x\rangle^{s-\delta+j} \langle D_x\rangle^{\delta-2j} D_t^j u)(t,\cdot)\|_{L^2(\mathbb{R}^d)}$$

and

$$\|u\|_{Y^{s,\delta}} = \sum_{j=0}^{s} \int_0^T \|(\langle \sigma D_x\rangle^{s-\delta+j} \langle D_x\rangle^{\delta-2j} D_t^j u)(t,\cdot)\|_{L^2(\mathbb{R}^d)},$$

respectively.

In view of these definitions, we automatically have the embedding $X^{s,\delta} \subseteq Y^{s,\delta}$. We also define the trace space as

$$H^{s,\delta}(\mathbb{R}^d) = X^{s,\delta}|_{t=0}.$$

This coincides with the 2-microlocal Sobolev with respect to the Langrangian $T_0^*(\mathbb{R}^d)$, see the Appendix.

Remark 3.38. Surely, another convenient definition would be the following: For parameters $s \in \mathbb{N}_0$, $\delta \in \mathbb{R}$, the space $\mathscr{H}^{s,\delta}((0,T) \times \mathbb{R}^d)$ consists of all functions $u\colon (0,T) \times \mathbb{R}^d \longrightarrow \mathbb{C}$ satisfying

$$\Lambda^{s-j,\delta-2j} D_t^j u \in L^2((0,T) \times \mathbb{R}^d), \qquad 0 \le j \le s.$$

For general $s \in \mathbb{R}$, $\delta \in \mathbb{R}$, the space $\mathscr{H}^{s,\delta}$ is then defined by interpolation and duality. This would be a huge advantage in contrast to the spaces $X^{s,\delta}$. However, with $X^{s,\delta}$ we do not have to worry about the traces at $t = 0$, because they continuously run into the above mentioned 2-microlocal spaces.

We are now going to prove the important mapping properties.

Proposition 3.39. *For $m \in \mathbb{N}_0$ we have*

$$\Lambda^{m,p} \in \mathscr{L}(X^{s+m,\delta+p}, X^{s,\delta}) \qquad \forall\,(s,\delta) \in \mathbb{N}_0 \times \mathbb{R}.$$

Proof. We use a similar reduction argument as in the previous proposition. Let $s \in \mathbb{N}_0$. Then we have to show, that

$$\Lambda^{s-j,\delta-2j} D_t^j(\Lambda^{m,p}u) \in \mathscr{C}([0,T], L^2(\mathbb{R}^d))$$

for $0 \le j \le s$, provided $u \in X^{s+m,\delta+p}$. By using commutators we have

$$D_t^j \Lambda^{m,p} = \Lambda^{m,p} D_t^j + [D_t^j, \Lambda^{m,p}],$$

and also

$$\Lambda^{s-j,\delta-2j} D_t^j(\Lambda^{m,p}u) = \Lambda^{s-j,\delta-2j}\Lambda^{m,p} D_t^j u + \Lambda^{s-j,\delta-2j}[D_t^j, \Lambda^{m,p}].$$

In the first term we get $\Lambda^{s-j,\delta-2j}\Lambda^{m,p} = \Lambda^{s+m-j,\delta+p-2j} + R$, where the remainder belongs to $\mathrm{Op}(\widetilde{\Sigma}^{s+m-j-1,\delta+p-2j})$. Hence,

$$\Lambda^{s-j,\delta-2j}\Lambda^{m,p} D_t^j u = \Lambda^{s+m-j,\delta+p-2j} D_t^j u + R D_t^j u.$$

The first summand belongs to $\mathscr{C}([0,T], L^2(\mathbb{R}^d))$ by assumption, whereas the second summand can be rewritten as

$$R D_t^j u = R(\Lambda^{s+m-j,p+\delta-2j})^{-1}\Lambda^{s+m-j,p+\delta-2j} D_t^j u$$

for all $0 \le j \le s$. Again, the function $\Lambda^{s+m-j,p+\delta-2j} D_t^j u$ belongs to $\mathscr{C}([0,T], L^2(\mathbb{R}^d))$ by assumption, whereas

$$R(\Lambda^{s+m-j,p+\delta-2j})^{-1} \in \mathrm{Op}(\widetilde{\Sigma}^{-1,0}) \subseteq \mathrm{Op}(\widetilde{\Sigma}^{0,0}),$$

which are $\mathscr{C}([0,T], L^2(\mathbb{R}^d))$-continuous, since $\mathrm{Op}(\widetilde{\Sigma}^{0,0}) \subseteq \mathrm{Op}(\Sigma^{0,0,\dagger})$ in view of Corollary 3.14.

Note that $\Lambda^{m,p} = \mathrm{Op}(\lambda^{m,p})$. By using iterated commutators, we further have

$$[D_t^j, \Lambda^{m,p}] = \sum_{k=1}^{j} \binom{j}{k} \mathrm{Op}(D_t^k \lambda) D_t^{j-k},$$

and so

$$\Lambda^{s-j,\delta-2j}[D_t^j, \Lambda^{m,p}]u \sim \sum_{k=1}^{j} \Lambda^{s+m-j,p+\delta+2k-2j} D_t^{j-k} u.$$

Inductively also these terms belong to $\mathscr{C}([0,T], L^2(\mathbb{R}^d))$. □

This leads to the following corollary:

Corollary 3.40. *For $m \in \mathbb{N}_0$ we have*

$$\mathrm{Op}(\Sigma^{m,p,\dagger}) \subset \mathscr{L}(X^{s+m,\delta+p}, X^{s,\delta}) \qquad \forall\,(s,\delta) \in \mathbb{N}_0 \times \mathbb{R}.$$

Let us now prove the following helpful lemma:

Lemma 3.41. *For $(s,\delta) \in \mathbb{N}_0 \times \mathbb{R}$ we have $D_t\colon X^{s+1,\delta} \longrightarrow X^{s,\delta-2}$ and*

$$u \in X^{s+1,\delta} \iff \Lambda^{1,0}u \in X^{s,\delta} \text{ and } \Lambda^{0,-2}D_t u \in X^{s,\delta}.$$

Proof. The first part is easily done by a simple index-shift. In fact, let $u \in X^{s+1,\delta}$, then

$$\Lambda^{s+1-j,\delta-2j} D_t^j u \in \mathscr{C}([0,T], L^2(\mathbb{R}^d))$$

for $0 \le j \le s+1$. Define $k := j-1$, then for $j \ne 0$ we have $0 \le k \le s$ and

$$\Lambda^{s-k,\delta-2k-2} D_t^{k+1} u \in \mathscr{C}([0,T], L^2(\mathbb{R}^d)),$$

which is $\Lambda^{s-k,\delta-2-2k} D_t^k(D_t u) \in \mathscr{C}([0,T], L^2(\mathbb{R}^d))$ and so $D_t u \in X^{s,\delta}$.

The second statement follows directly by the mapping properties of D_t and $\Lambda^{m,p}$. $\square$

With the same arguments we finally get the following similar results:

Corollary 3.42. *Let $(s,\delta) \in \mathbb{N}_0 \times \mathbb{R}$. For $m \in \mathbb{N}_0$ we have*

$$\mathrm{Op}(\widetilde{\Sigma}^{m,p}) \in \mathscr{L}(Y^{s+m,\delta+p}, Y^{s,\delta}).$$

Moreover, if $A(t,x,D_x) \in \mathrm{Op}(\widetilde{\Sigma}^{m,p})$, then

$$A(0,x,D_x) \in \mathscr{L}(H^{s+m,\delta+p}, H^{s,\delta}).$$

Remark 3.43. Compared to the results in [**Bon**], the space $\mathscr{C}H^{s,\delta}|t=0$ corresponds to the weighted Sobolev $SP(\delta, s-\delta)$, while $\Lambda^{m,p}(0,x,D_x)$ belongs to $\mathrm{Op}(\Sigma_0^{p,m-p})$, see the equivalence in (A.1).

Next, we now want to prove partial hypoellipticity. For $r \in \mathbb{N}_0$, we also set

$$X^{s,r,\delta} := \{u \,|\, \partial_t^j u \in \mathscr{C}H^{s-j,\delta-2j},\ 0 \le j \le r\}$$
$$Y^{s,r,\delta} := \{f \,|\, \partial_t^j f \in L^1 H^{s-j,\delta-2j},\ 0 \le j \le r\}.$$

Note that $X^{s,s,\delta} = X^{s,\delta}$ for $s \in \mathbb{N}_0$ and $X^{s,0,\delta} = \mathscr{C}H^{s,\delta}$. Then the following result is valid:

Theorem 3.44. *Let $r \in \mathbb{N}_0$, $u \in \mathscr{C}H^{s,\delta}$, $f \in Y^{s,r,\delta}$ and $D_t u = a(t,x,D_x)u + f$, where $a(t,x,D_x) \in \operatorname{Op}(\widetilde{\Sigma}^{1,2})$. Then $u \in X^{s,r,\delta}$.*

Proof. We prove this statement by an induction over $r \in \mathbb{N}_0$. If $r = 0$, then nothing is to do. Assume that this statement is valid for a certain $r \in \mathbb{N}_0$ and $f \in Y^{s,r+1,\delta}$. Then we have to show, that $D_t^{r+1}u \in \mathscr{C}H^{s-r-1,\delta-2r-2}$. Using the differential equation leads us to

$$D_t^{r+1}u = D_t^r(D_t u) = D_t^r(a(t,x,D_x)u + f) = D_t^r a(t,x,D_x)u + D_t^r f,$$

so we have to compute the operator $D_t^r a(t,x,D_x)$. Using iterated commutators, we get again

$$D_t^r a(t,x,D_x) = \sum_{j=0}^{r} \binom{r}{j} (D_t^j a)(t,x,D_x) D_t^{r-j}.$$

Thus,

$$D_t^{r+1}u = a(t,x,D_x)D_t^r u + \sum_{j=1}^{r} \binom{r}{j} (D_t^j a)(t,x,D_x) D_t^{r-j} u + D_t^r f.$$

First, we have $D_t^r u \in \mathscr{C}H^{s-r,\delta-2r}$, and by mapping properties also

$$a(t,x,D_x)D_t^r u \in \mathscr{C}H^{s-r-1,\delta-2r-2}.$$

Since $(D_t^j a)(t,x,D_x) \in \operatorname{Op}(\widetilde{\Sigma}^{1,2+2j})$ and $D_t^{r-j}u \in \mathscr{C}H^{s-r+j,\delta-2r+2j}$, we also obtain

$$(D_t^j a)(t,x,D_x)D_t^{r-j}u \in \mathscr{C}H^{s-r-1+j,\delta-2r-2}, \qquad 1 \le j \le r,$$

which is even better than $\mathscr{C}H^{s-r-1,\delta-2r-2}$. If $f \in Y^{s,r+1,\delta}$, so

$$D_t^{r+1} f \in L^1 H^{s-r-1,\delta-2r-2}.$$

With the identity

$$f(x) = f(0) + \int_0^x (\partial_t^r f)(s)\, ds$$

we have $D_t^r f \in \mathscr{C}H^{s-r-1,\delta-2r-2}$. This finally yields $D_t^{r+1}u \in \mathscr{C}H^{s-r-1,\delta-2r-2}$, and so $u \in X^{s,r+1,\delta}$. □

As a consequence we obtain:

Corollary 3.45. *In the proof of the main theorem it suffices to consider $r = 0$.*

4 Proof of the Main Theorem

In this Chapter we are going to prove the main theorem. We first define the following expression:

Definition 4.1. We define

$$\Phi(A_{00}, A_{01}, M_{00}, M_{01}) := \left(MAM^{-1} + (D_t M)M^{-1}\right)_{01}.$$

Let us now state the main theorem with the notation we got so far.

Theorem 4.2. *Let $A \in \mathrm{Op}(\widetilde{\Sigma}^{1,2})$. Assume there exists a $M_0 \in \Sigma^{(0),0}$ with $|\det M_0| \gtrsim 1$, $M_0 A_0 M_0^{-1}$ is Hermitian, such that*

$$I - 2x\left(M_{00}A_{00}M_{00}^{-1}\right)_\xi > 0,$$

where $M_{00} \in \Sigma^{(0,0)}$ is the $(0,0)$-bihomogeneous component of M_0.

Then for every M_{01} there exists a $\delta_0 = \delta_0(A_{00}, A_{01}) \in \mathbb{R}$ such that

$$\mathrm{Im}\left(\Phi(A_{00}, A_{01}, M_{00}, M_{01})\right) \leq \frac{\delta_0}{2r^2}\left(I - 2x\left(M_{00}A_{00}M_{00}^{-1}\right)_\xi\right), \tag{4.1}$$

and a $\gamma_0 = \gamma(A_{00})$, with the property that for all $s \in \mathbb{N}_0$, $\delta \geq \delta_0 + s\gamma_0$, $U_0 \in H^{s,\delta}$, $F \in Y^{s,\delta}$, the Cauchy problem (1.1) possesses a unique solution $U \in X^{s,\delta}$. Moreover, the a priori estimate

$$\|U\|_{X^{s,\delta}} \lesssim \|U_0\|_{H^{s,\delta}} + \|F\|_{Y^{s,\delta}}$$

is valid.

For the meaning of Φ, see Definition 4.4. The relation between δ_0, γ_0 and A_{00}, A_{01} is quite complicated. We will provide upper bounds on both δ_0 and γ_0 in Section 4.3.

In fact, the operator A is not assumed to have Hermitian principal part, but we assume the existence of a symmetrizer, see Definition 4.6. With this operator we can formulate an equivalent system, whose principal symbol is Hermitian. The second reduction is, that we can "shift" the parameter δ to 0.

Note that δ depends on s and increases as $s \to \infty$. To prove an energy estimate for every $s \in \mathbb{N}_0$, we will us an inductive argument. By using the spectral shift $\delta \to 0$, we will assume, that the energy estimate holds for an $s \in \mathbb{N}_0$ with $\delta = 0$. We then show, that with the step $s \to s+1$, also the spectral parameter δ increases by γ_0. For the next step in the induction, we again shift the new δ to 0.

In the sequel, we will consider operators of the form $KAK^{-1} + (D_t K)K^{-1}$, where $A \in \mathrm{Op}(\widetilde{\Sigma}^{1,2})$, $K \in \mathrm{Op}(\widetilde{\Sigma}^{m,p})$ and K is invertible. Using the composition rules, see Proposition 3.30, we can formulate the following technical lemma:

Lemma 4.3. *We have*

$$
\begin{aligned}
&\left(MAM^{-1} + (D_t M)M^{-1}\right)_{01} \\
&\quad = M_{00}\left[(M_{00}^{-1}M_{01}, A_{00}\right](M_{00})^{-1} + M_{00}A_{01}(M_{00})^{-1} - i(M_{00})_\xi (A_{00})_x (M_{00})^{-1} \\
&\qquad - i(M_{00})_t (M_{00})^{-1} \\
&\qquad - i\left(M_{00}A_{00}(M_{00})^{-1}(M_{00})_\xi - (M_{00})_\xi A_{00} - M_{00}(A_{00})_\xi\right)(M_{00})^{-1}(M_{00})_x (M_{00})^{-1}.
\end{aligned}
$$

When M is scalar with $M_{01} = 0$, then this simplifies to (see last summand):

$$
\begin{aligned}
\left(MAM^{-1} + (D_t M)M^{-1}\right)_{01} = A_{01} &- i(M_{00})_\xi (A_{00})_x (M_{00})^{-1} + i(M_{00})^{-1}(A_{00})_\xi (M_{00})_x \\
&- i(M_{00})_t (M_{00})^{-1}.
\end{aligned}
$$

Consider now $M = \Lambda^{m,p}$. This gives us

$$
\left(\Lambda A \Lambda^{-1} + (D_t \Lambda)\Lambda^{-1}\right)_{01} = A_{01} - \frac{im\xi}{|\xi|^2}(A_{00})_x + \frac{i(m-p)x}{r^2}(A_{00})_\xi - \frac{i(m-p)}{2r^2} I.
$$

If additionally $m = 0$ then

$$
\left(\Lambda A \Lambda^{-1} + (D_t \Lambda)\Lambda^{-1}\right)_{01} = A_{01} + \frac{i(m-p)x}{r^2}(A_{00})_\xi - \frac{i(m-p)}{2r^2} I.
$$

In this case we also have

$$
\left(\Lambda A \Lambda^{-1}\right)_{01} = A_{01} + \frac{i(m-p)x}{r^2}(A_{00})_\xi .
$$

We end this short overview with giving the precise formulation for Φ.

Corollary 4.4. *We have*

$$\begin{aligned}
&\Phi(A_{00}, A_{01}, M_{00}, M_{01}) \\
&\quad = M_{00}\left[(M_{00}^{-1}M_{01}, A_{00}\right](M_{00})^{-1} + M_{00}A_{01}(M_{00})^{-1} - i(M_{00})_\xi(A_{00})_x(M_{00})^{-1} \\
&\qquad - i(M_{00})_t(M_{00})^{-1} \\
&\qquad - i\left(M_{00}A_{00}(M_{00})^{-1}(M_{00})_\xi - (M_{00})_\xi A_{00} - M_{00}(A_{00})_\xi\right)(M_{00})^{-1}(M_{00})_x(M_{00})^{-1}.
\end{aligned}$$

Note that $\Phi(A_{00}, A_{01}, I, 0) = A_{01}$.

4.1 Basic $\mathscr{C}L^2$-energy estimate

Let us derive the basic energy estimate, which later will give us the a priori estimate in $\mathscr{C}L^2$. For this, we do not need any special structure on A in terms of principal symbols. However, in the following theorem we see the influence of the remainder class $\Sigma^{m-2,p,\dagger} + \Sigma^{m-1,p-1,\dagger}$ for $(m,p) = (1,2)$.

Theorem 4.5. *Let $A \in \mathrm{Op}(\Sigma^{1,2,\dagger})$ be an operator, such that $\mathrm{Im}(A) \in \mathrm{Op}(\Sigma^{0,2,\dagger})$ and*

$$\mathrm{Im}(A(t,x,\xi)) \lesssim \left(\frac{\langle\xi\rangle}{\langle\sigma\xi\rangle} + \frac{\langle\xi\rangle^2}{\langle\sigma\xi\rangle^3}\right) I$$

Then each solution $U = U(t,x) \in \mathscr{C}L^2$ to the problem (1.1) with $U_0 \in L^2(\mathbb{R}^d)$ and $F \in L^1L^2$ satisfies the a priori estimate

$$\|U\|_{\mathscr{C}L^2} \lesssim \|U_0\|_{L^2} + \|F\|_{L^1L^2}.$$

Proof. Let us write $q(t,x,\xi) = \langle\xi\rangle\langle\sigma\xi\rangle^{-1} + \langle\xi\rangle^2\langle\sigma\xi\rangle^{-3}$. We assume, that the operator $D_t - A$ possesses a forward fundamental solution $X(t,t')\colon \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$, i.e.

$$\begin{cases} (D_t - A(t,x,D_x))X(t,t') = 0, & 0 \le t' \le t \le T, \\ X(t',t') = I, & . \end{cases}$$

Then we can represent the solution U via

$$U(t,x) = X(t,0)U_0(x) + i\int_0^T X(t,t')F(t',x)dt'$$

and so have to prove an uniform estimate of the form

$$\|X(t,t')V\|_{L^2(\mathbb{R}^d)} \lesssim \|V\|_{L^2(\mathbb{R}^d)}, \tag{4.2}$$

for all $0 \leq t' \leq t \leq T$ and $V \in \mathscr{S}(\mathbb{R}^d)$. Together with the representation for U we would get

$$\|U(t,\cdot)\|_{L^2(\mathbb{R}^d)} \lesssim \|U_0\|_{L^2(\mathbb{R}^d)} + \int_0^t \|F(t',\cdot)\|_{L^2(\mathbb{R}^d)}\,dt'$$

for all $0 \leq t \leq T$ and further

$$\begin{aligned}\|U\|_{\mathscr{C}L^2} &= \sup_{t\in[0,T]} \|U(t,\cdot)\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \|U_0\|_{L^2(\mathbb{R}^d)} + \int_0^T \|F(t',\cdot)\|_{L^2(\mathbb{R}^d)}\,dt' \\ &\lesssim \|U_0\|_{L^2} + \|F\|_{L^1L^2}.\end{aligned}$$

We are now going to show (4.2). Consider first the primitive

$$p(t,x,\xi) = \int_0^t q(t,x,\xi)\,ds = \int_0^t \frac{\langle\xi\rangle}{\langle\sigma\xi\rangle}\,ds + \int_0^t \frac{\langle\xi\rangle^2}{\langle\sigma\xi\rangle^3}\,ds = I_1 + I_2,$$

where $\sigma(s,x) = \sqrt{s+|x|^2}$. Note that there are constants $c, C > 0$ such that

$$c\langle\sigma\xi\rangle \leq \langle\sigma\langle\xi\rangle\rangle \leq C\langle\sigma\xi\rangle.$$

Then we calculate, by change of variables, i.e. $t(s) = \langle\sigma\langle\xi\rangle\rangle^2$,

$$\begin{aligned}I_1 &= \langle\xi\rangle \int_0^t \frac{ds}{\langle\sigma\xi\rangle} \lesssim \langle\xi\rangle \int_0^t \frac{ds}{\langle\sigma\langle\xi\rangle\rangle} = \langle\xi\rangle^{-1} \int_{\langle|x|\langle\xi\rangle\rangle^2}^{\langle\sigma\langle\xi\rangle\rangle^2} \frac{d\tau}{\sqrt{\tau}} \\ &\lesssim \langle\xi\rangle^{-1}\left[\sqrt{\tau}\right]_{\langle|x|\langle\xi\rangle\rangle^2}^{\langle\sigma\langle\xi\rangle\rangle^2} \lesssim \langle\sigma\langle\xi\rangle\rangle\langle\xi\rangle^{-1} \lesssim \langle\xi\rangle^{-1}\langle\sigma\xi\rangle \in \Sigma^{0,-1,\dagger}\end{aligned}$$

and similarly

$$\begin{aligned}I_2 &= \langle\xi\rangle^2 \int_0^t \frac{ds}{\langle\sigma\xi\rangle^3} \lesssim \langle\xi\rangle^2 \int_0^t \frac{ds}{\langle\sigma\langle\xi\rangle\rangle^3} = \int_{\langle|x|\langle\xi\rangle\rangle^2}^{\langle\sigma\langle\xi\rangle\rangle^2} \frac{d\tau}{\sqrt{\tau}^3} \\ &\lesssim \left[\frac{1}{\sqrt{\tau}}\right]_{\langle|x|\langle\xi\rangle\rangle^2}^{\langle\sigma\langle\xi\rangle\rangle^2} \lesssim \langle\sigma\langle\xi\rangle\rangle^{-1} \lesssim \langle\sigma\xi\rangle^{-1} \in \Sigma^{-1,0,\dagger}\end{aligned}$$

Thus,

$$p(t,x,\xi) \in \Sigma^{0,-1,\dagger} + \Sigma^{-1,0,\dagger} \subseteq \Sigma^{0,0,\dagger}.$$

For $0 \leq t' \leq t \leq T$, we define $Y(t,t') \colon \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}(\mathbb{R}^d)$ via

$$Y(t,t') = e^{-p(t,x,D_x)}e^{p(t',x,D_x)}X(t,t').$$

Here the operators $e^{\pm p(t,x,D_x)}$ are of order zero and invertible. Furthermore, we have by construction $Y(t',t') = I$. Then

$$\begin{aligned}\partial_t Y(t,t') &= -q(t,x,D_x)Y(t,t') + ie^{-p(t,x,D_x)}e^{p(t',x,D_x)}A(t,x,D_x)X(t,t') \\ &= B(t,x,D_x)Y(t,t')\end{aligned}$$

where

$$B(t,x,D_x) = iA - qI + \left[e^{-p(t,x,D_x)}e^{p(t',x,D_x)}, iA\right] e^{-p(t',x,D_x)}e^{p(t,x,D_x)}.$$

For B we have $B(t,x,\xi) \in \Sigma^{1,2,\dagger}$ and $\mathrm{Re}(B) \le K$ for a constant $K > 0$. Thus, Garding's inequality gives us

$$\begin{aligned}\partial_t \|Y(t,t')V\|^2_{L^2(\mathbb{R}^d)} &= 2\,\mathrm{Re}(\partial_t Y(t,t')V, Y(t,t')V) = 2\,\mathrm{Re}(BY(t,t')V, Y(t,t')V)\\ &\lesssim \|Y(t,t')V\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Finally using Gronwall's lemma, we obtain

$$\|Y(t,t')V\|^2_{L^2(\mathbb{R}^d)} \lesssim \|Y(t',t')V\|^2_{L^2(\mathbb{R}^d)} \lesssim \|V\|^2_{L^2(\mathbb{R}^d)}.$$

This gives us (4.2), since the operators $e^{\pm p(t,x,D_x)}$ are isomorphisms on $L^2(\mathbb{R}^d)$. $\square$

Later, this estimate basically will give us the $\mathscr{C}L^2$-wellposedness.

4.2 Reductions

Let us now come to the reductions. First we symmetrize the principal part of the system.

Definition 4.6. The system (1.1) is called symmetrizable hyperbolic if there is an $M_0 \in \Sigma^{(0),0}$ such that $(M_0 A_0 M_0^{-1})(t,x,\xi)$ is Hermitian for all $(t,x,\xi) \in \Upsilon \times \dot{\mathbb{R}}^d$.

We will now prove the existence of such a symmetrizer M.

Theorem 4.7. *Let* $(M_0, M_{01}) \in \Sigma^{(0),0} \times \Sigma^{(-1,0)}$ *with* $|\det M_0| \gtrsim 1$. *Then there exists a* $M \in \mathrm{Op}(\widetilde{\Sigma}^{0,0})$ *with*

$$\sigma^0_\Psi(M) = M_0 \qquad \textit{and} \qquad \sigma^{-1,0}_{\Psi,d}(M) = M_{01}.$$

Moreover, M *possesses a parametrix.*

Proof. Given $(M_0, M_{01}) \in \Sigma^{(0),0} \times \Sigma^{(-1,0)}$ the existence of this operator follows directly from the exactness of the sequence in Lemma 3.28. The ellipticity condition $|\det M_0| \gtrsim 1$ implies, that M_0 is invertible with inverse in $\Sigma^{(0),0}$. Again with Lemma 3.28 there exists $Q \in \mathrm{Op}(\widetilde{\Sigma}^{0,0})$ with $\sigma^0_\Psi(Q) = M_0^{-1}$. Moreover,

$$\sigma^0_\Psi(MQ - I) = \sigma^0_\Psi(M)\sigma^0_\Psi(Q) - \sigma^0_\Psi(I) = 0,$$

so $R := I - MQ \in \mathrm{Op}(\widetilde{\Sigma}^{-1,0})$. Thus Q is a right parametrix of order 1. Let now $Q_k = Q(I + R + \cdots R^{k-1})$, then

$$MQ_k = (I - R)(I + R + \cdots R^{k-1}) = I - R^k$$

with $R^k \in \mathrm{Op}(\widetilde{\Sigma}^{-k,0})$, so Q_k is a right parametrix of order k. By the same procedure we get a left parametrix Q_k' of order k. Evaluating $Q_k' P Q_k$ in two ways one obtains that $Q_k - Q_k' \in \mathrm{Op}(\widetilde{\Sigma}^{-k,0})$, and from this that Q_k is also a left parametrix of order k. Iterating this process and using the asymptotic completeness from Proposition 3.8, we imply that M has a parametrix of order $-\infty$. $\square$

We can refine this result by the following statement.

Theorem 4.8. *Let* $(M_0, M_{01}) \in \Sigma^{(0),0} \times \Sigma^{(-1,0)}$ *with* $|\det M_0| \gtrsim 1$. *Then there exists a* $M \in \mathrm{Op}(\widetilde{\Sigma}^{0,0})$ *invertible with*

$$\sigma_\Psi^0(M) = M_0 \qquad \textit{and} \qquad \sigma_{\Psi,d}^{-1,0}(M) = M_{01}.$$

Proof. We still have to prove the invertibility and shall adopt the strategy of Theorem 3.32. Therefore, for $K \geq 1$, we construct a parameter-dependent family of symbols $M_K \in \widetilde{\Sigma}^{0,0}$ satisfying

$$|\partial_t^j \partial_x^\alpha \partial_\xi^\beta M_K| \lesssim \langle \xi \rangle_K^{2j - |\beta| + |\alpha|} \langle \sigma\xi \rangle_K^{|\alpha| - 2j}.$$

To do so, we extend the definition of the class $\Sigma^{(m),p}$ to $d+1$ covariables. Thus we consider amplitude functions $a \in \mathscr{C}^\infty(\Upsilon, S^{(m)}(\dot{\mathbb{R}}^{d+1}))$, which admit an asymptotic expansion of the form

$$a(t, x, \xi') \sim \sum_{k \geq 0} \sigma^{m-p+k} b_k(\vartheta, \xi') \quad \text{as } r \to +0$$

with $b_k \in \mathscr{C}^\infty(\Theta, S^{(m)}(\dot{\mathbb{R}}^{d+1}))$. Given such a function a and setting $\xi' \longmapsto (\xi, K)$ provides a K-dependent family of symbols in $\Sigma^{(m),p}$, also homogeneous of order m in K. Vice versa, given a function f, living on $\mathbb{S}^{d-1}$, it is possible to extend it homogeneously to a function $\mathbb{S}_+^d$. Thus, starting with a symbol $a \in \mathscr{C}^\infty(\Upsilon, S^{(m)}(\dot{\mathbb{R}}^d))$ it is possible to extend it to a symbol in $\mathscr{C}^\infty(\Upsilon, S^{(m)}(\dot{\mathbb{R}}^{d+1}))$. The construction of $M_{01,K}$ is similar.

For given $(M_0, M_{01}) \in \Sigma^{(0),0} \times \Sigma^{(-1,0)}$ we use this construction to obtain the K-dependent symbols $M_{0,K}$ and $M_{01,K}$, respectively. For $K \geq 1$ define

$$M_K := \chi^+(M_{0,K} + M_{01,K}).$$

Then each symbol M_K belongs to $\widetilde{\Sigma}^{0,0}$. Similarly by ellipticity M_K^{-1} is in $\widetilde{\Sigma}^{0,0}$. Define

$$R_K := M_K\#(M_K)^{-1} - M_K(M_K)^{-1} = M_K\#(M_K)^{-1} - 1.$$

For any $\ell \in \mathbb{N}_0$ let $(j,\alpha,\beta) \in \mathbb{N}_0^{1+2d}$ with $j+|\alpha|+|\beta| \le \ell$. We want to show

$$|R_K|_{0,0;\ell} \le \frac{C_\ell}{K},$$

where

$$|R_K|_{0,0;\ell} = \sup_{\substack{(t,x,\xi)\in[0,T]\times\mathbb{R}^{2d}\\ j+|\alpha|+|\beta|\le\ell}} \langle\xi\rangle^{2j+|\beta|-|\alpha|}\langle\sigma\xi\rangle^{|\alpha|+2j}|\partial_t^j\partial_x^\alpha\partial_\xi^\beta R_K|.$$

We will prove $R_K \longrightarrow 0$ in $\Sigma^{0,0}$ for $K \longrightarrow \infty$. By Leibniz rule, it is

$$|\partial_t^j\partial_x^\alpha\partial_\xi^\beta R_K| \le \sum_{|\gamma|\ge1}\frac{1}{\gamma!}\sum_{j'+|\alpha'|+|\beta'|\le\ell} c_{j'\alpha'\beta'}|\partial_t^{j'}\partial_x^{\alpha'}\partial_\xi^{\beta'+\gamma}M_K||\partial_t^{j-j'}\partial_x^{\alpha-\alpha'+\gamma}\partial_\xi^{\beta-\beta'}(M_K)^{-1}|.$$

With the same estimates as in Theorem 3.32 we get

$$|R_K|_{0,0;\ell} \le C_\ell \sup_{(t,x,\xi)\in[0,T]\times\mathbb{R}^{2d}} \langle\sigma\xi\rangle_K^{-1} = \frac{C_\ell}{K}.$$

Hence, $R_K \longrightarrow 0$ in $\Sigma^{0,0}$ for $K \longrightarrow \infty$. This implies $R_K(t,x,D_x) \longrightarrow 0$ in $\mathscr{L}(L^2((0,T)\times\mathbb{R}^d)$ for $K \longrightarrow \infty$. Thus $\mathrm{Op}(M_K^{-1}) \circ (1+R_K)^{-1}$ is a right inverse to $\mathrm{Op}(M_K)$ for large $K \ge 1$. In a similar way, a left inverse to $\mathrm{Op}(M_K)$ is contructed. $\square$

This enables us to state the following Theorem.

Theorem 4.9. *In the proof of the main theorem we can assume $A_0 = A_0^*$.*

Proof. By the mapping properties we have, that the symmetrizer $M \in \mathrm{Op}(\widetilde{\Sigma}^{0,0})$ is an isomorphism from $\mathscr{C}H^{s,\delta}$ to $\mathscr{C}H^{s,\delta}$, while $M(0,x,D_x)$ is an isomorphism from $H^{s,\delta}$ into itself. We now set $V := MU$ and consider the equivalent system

$$\begin{cases} D_tV &= B(t,x,D_x)V + G(t,x), \quad \text{in } (0,T)\times\mathbb{R}^d,\\ V(0,x) &= V_0(x), \qquad\qquad\qquad\quad \text{on } \mathbb{R}^d. \end{cases}$$

where $B = MAM^{-1} + (D_tM)M^{-1}$, $V_0 = M(0,x,D_x)U_0$ and $G = MF$. We remark, that $MAM^{-1} \in \mathrm{Op}(\widetilde{\Sigma}^{1,2})$, whereas $(D_tM)M^{-1} \in \mathrm{Op}(\widetilde{\Sigma}^{0,2})$. According to the composition rules, we get $B \in \mathrm{Op}(\widetilde{\Sigma}^{1,2})$ and

$$B_0 = \sigma_\Psi^1(B) = \sigma_\Psi^1(MAM^{-1}) = M_0A_0M_0^{-1},$$

which is Hermititan for all $(t,x,\xi) \in \Upsilon \times \dot{\mathbb{R}}^d$. Consequently,

$$B_{01} = \Phi(A_{00}, A_{01}, M_{00}, M_{01}),$$

and so condition (4.1) turns into

$$\mathrm{Im}(B_{01}) \leq \frac{\delta}{2r^2}\left(I - 2x(B_{00})_\xi\right).$$

This completes the proof. Here, δ has not changed. □

Therefore, we can assume $M_0 = I$ and $M_{01} = 0$. The main ingredient of the proof of the basic $\mathscr{C}L^2$-energy estimate was the upper bound for $\mathrm{Im}(A(t,x,\xi))$. In general this bound does not hold. Fortunately, we still can prove the following theorem:

Theorem 4.10. *Let $A \in \mathrm{Op}(\widetilde{\Sigma}^{1,2})$, $A_0 = A_0^*$ and*

$$\mathrm{Im}(A_{01}) \leq \frac{\delta}{2r^2}\left(I - 2x(A_{00})_\xi\right).$$

Then, if $F \in L^1H^{0,\delta}$ and $U_0 \in H^{0,\delta}(\mathbb{R}^d)$, we have an energy estimate for a solution $U \in \mathscr{C}H^{0,\delta}$ of the form

$$\|U\|_{\mathscr{C}H^{0,\delta}} \lesssim \|U_0\|_{H^{0,\delta}} + \|F\|_{L^1H^{0,\delta}}.$$

Proof. Since $s = 0$, it is $\delta \geq \delta_0$. Consider the operator $\Lambda = \Lambda^{0,-\delta} \in \mathrm{Op}(\widetilde{\Sigma}^{0,-\delta})$ abd set $V = \Lambda U$. Then the Cauchy problem is equivalent to

$$\begin{cases} D_t V \quad = (\Lambda A \Lambda^{-1} + (D_t\Lambda)\Lambda^{-1})V + \Lambda F, \\ V(0,x) = \Lambda(0,x,D_x)U_0(x) = V_0, \end{cases}$$

with $\Gamma := \Lambda A \Lambda^{-1} + (D_t\Lambda)\Lambda^{-1} \in \mathrm{Op}(\widetilde{\Sigma}^{1,2})$, and on the level of principal part we obtain $\sigma_\Psi^1(\Lambda A \Lambda^{-1}) = \sigma_\Psi^1(A)$, since Λ is scalar.

Again we have to compute $\Phi(A_{00}, A_{01}, \Lambda_{00}, \Lambda_{01})$. By having $\Lambda_0 = \sigma^\delta$, $\Lambda_{00} = r^\delta$ and $\Lambda_{01} = 0$, this expression simplifies to

$$\Phi(A_{00}, A_{01}, \Lambda_{00}, \Lambda_{01}) = A_{01} - \frac{i\delta}{2}r^{-2} + i\delta r^{-2}x(A_{00})_\xi.$$

Hence,

$$\mathrm{Im}\left(\Phi(A_{00}, A_{01}, \Lambda_{00}, \Lambda_{01})\right) = \mathrm{Im}(A_{01}) - \frac{\delta}{2r^2}\left(I - 2x(A_{00})_\xi\right) \leq 0,$$

by assumption. Thus, the operator Γ satisfies the conditions of Theorem 4.5, and so we obtain

$$\|V\|_{\mathscr{C}L^2} \lesssim \|V_0\|_{L^2} + \|\Lambda F\|_{L^1L^2}.$$

By using Λ as an isomorphism from $Z^{0,0} \longrightarrow Z^{0,\delta}$ for $Z \in \{X, Y, H\}$, we finally have

$$\|U\|_{\mathscr{C}H^{0,\delta}} \lesssim \|U\|_{H^{0,\delta}} + \|F\|_{L^1H^{0,\delta}}.$$

This completes the proof. □

4.3 Proof of Theorem 1.1

We now come to the proof of the main result. We split this proof into three steps. Therefore, we always assume the reductions made in the previous section, which are

$$A_0^* = A_0 \quad \text{and} \quad \mathrm{Im}(A_{01}) \leq 0.$$

Step 1: Basic a a priori estimate. Each solution to U to the system (1.1) satisfies the a priori estimate

$$\|U\|_{\mathscr{C}L^2} \lesssim \|U_0\|_{L^2} + \|F\|_{L^1L^2}$$

Proof. Write

$$A(t,x,\xi) = \chi^+(t,x,\xi)\left(A_0(t,x,\xi) + A_1(t,x,\xi)\right) + A_r(t,x,\xi).$$

Then

$$\mathrm{Im}(A) = \chi^+\left(\mathrm{Im}(A_0) + \mathrm{Im}(A_1)\right) + \mathrm{Im}(A_r),$$

and since $\mathrm{Im}(A_0) = 0$ and $\mathrm{Im}(A_1) \leq 0$, we have

$$\mathrm{Im}(A) \lesssim \mathrm{Im}(A_r) \in \Sigma^{m-2,p,\dagger} + \Sigma^{m-1,p-1,\dagger}.$$

Hence, the statement follows immediately from Theorem 4.5. □

Step 2: Higher-order a priori estimate. Each solution $U \in \mathscr{C}H^{s,\delta}$ to the system (1.1) satisfies the a priori estimate

$$\|U\|_{\mathscr{C}H^{s,\delta}} \leq C\left(\|U_0\|_{H^{s,\delta}} + \|F\|_{L^1H^{s,\delta}}\right)$$

for $s \in \mathbb{N}_0$ and $\delta \geq \delta_0 + s\gamma_0$.

Proof. By the second reduction, we can consider the equivalent system with $\delta = 0$. We now use an induction on $s \in \mathbb{N}_0$. Note that after passing from s to $s+1$, the spectral parameter has to change.

In view of Lemma 3.41 we have $u \in \mathscr{C}H^{s+1,0}$ if and only if $\Lambda^{1,0}u \in \mathscr{C}H^{s,0}$ and $\Lambda^{0,-2}D_t u \in \mathscr{C}H^{s,0}$. If we define the vector

$$V := (\Lambda^{1,0}u, \Lambda^{0,-2}D_t u) \in \mathscr{C}H^{s,0},$$

then V is a solution of the Cauchy problem

$$\begin{cases} D_t \begin{pmatrix} \Lambda^{1,0}u \\ \Lambda^{0,-2}D_t u \end{pmatrix} & = \begin{pmatrix} P & 0 \\ Q & R \end{pmatrix} \begin{pmatrix} \Lambda^{1,0}u \\ \Lambda^{0,-2}D_t u \end{pmatrix} + \begin{pmatrix} \Lambda^{1,0}F \\ D_t\Lambda^{0,-2}F \end{pmatrix}, \\ \begin{pmatrix} \Lambda^{1,0}U \\ \Lambda^{0,-2}D_t U \end{pmatrix}(0,x) & = \begin{pmatrix} \langle xD_x\rangle U_0 \\ \langle D_x\rangle^{-2}\langle xD_x\rangle^2(A(0,x,D_x)U_0 + F(0,x)) \end{pmatrix}. \end{cases}$$

where

$$\mathscr{A} := \begin{pmatrix} P & 0 \\ Q & R \end{pmatrix} = \begin{pmatrix} \Lambda^{1,0}A(\Lambda^{1,0})^{-1} + (D_t\Lambda^{1,0})(\Lambda^{1,0})^{-1} & 0 \\ [(D_t\Lambda^{0,-2})A + \Lambda^{0,-2}(D_tA)](\Lambda^{1,0})^{-1} & \Lambda^{0,-2}A(\Lambda^{0,-2})^{-1} \end{pmatrix}.$$

In particular, $\mathscr{A}$ fulfills the differential equation, but for an $(2N)\times(2N)$ matrix. Let us now compute the corresponding principal symbols. We have $\Lambda^{1,0}A(\Lambda^{1,0})^{-1} \in \mathrm{Op}(\widetilde{\Sigma}^{1,2})$ and $(D_t\Lambda^{1,0})(\Lambda^{1,0})^{-1} \in \mathrm{Op}(\widetilde{\Sigma}^{0,2})$. Using the principal symbol formulas we obtain

$$\sigma^1_\Psi(P) = \sigma^1_\Psi(A)$$

and

$$\sigma^{0,2}_{\Psi,d}(P) = \sigma^{0,2}_{\Psi,d}(A) - i\frac{\xi(A_{00})_x}{|\xi|^2} - \frac{i}{2r^2} + i\frac{x(A_{00})_\xi}{r^2}.$$

Furthermore, $(D_t\Lambda^{0,-2})A(\Lambda^{1,0})^{-1}, \Lambda^{0,-2}(D_tA)(\Lambda^{1,0})^{-1} \in \mathrm{Op}(\widetilde{\Sigma}^{0,2})$, so we obtain

$$\sigma^{0,2}_{\Psi,d}((D_t\Lambda^{0,-2})A(\Lambda^{1,0})^{-1}) = -i(A_{00})|\xi|^{-1}r^{-1}$$

and

$$\sigma^{0,2}_{\Psi,d}(\Lambda^{0,-2}(D_tA)(\Lambda^{1,0})^{-1}) = -i(A_{00})_t|\xi|^{-1}r.$$

Thus,

$$\sigma^{0,2}_{\Psi,d}(Q) = -i(A_{00})|\xi|^{-1}r^{-1} - i(A_{00})_t|\xi|^{-1}r.$$

Last, we have $R \in \mathrm{Op}(\widetilde{\Sigma}^{1,2})$ with

$$\sigma^1_\Psi(R) = \sigma^1_\Psi(A) \text{ and } \sigma^{0,2}_{\Psi,d}(R) = \sigma^{0,2}_{\Psi,d}(A) + \frac{2ix}{r^2}(A_{00})_\xi.$$

Thus, we finally obtain

$$\sigma_\Psi^1(\mathscr{A}) = \begin{pmatrix} \sigma_\Psi^1(A) & 0 \\ 0 & \sigma_\Psi^1(A) \end{pmatrix}$$

and

$$\Phi(\mathscr{A}_{00}, \mathscr{A}_{01}, I, 0) = \begin{pmatrix} \Phi(A_{00}, A_{01}, I, 0) & 0 \\ 0 & \Phi(A_{00}, A_{01}, I, 0) \end{pmatrix} + \frac{i}{2r^2}\begin{pmatrix} \hat{a} & 0 \\ \hat{c} & \hat{d} \end{pmatrix}$$

where

$$\hat{a} = 2x(A_{00})_\xi - \frac{2r^2}{|\xi|^2}\xi(A_{00})_x - I, \ \ \hat{c} = -\frac{2r}{|\xi|}A_{00} - \frac{2r^3}{|\xi|}(A_{00})_t, \ \ \hat{d} = 4x(A_{00})_\xi.$$

Note that $A_{00} \in \Sigma^{(1,2)}$, so by applying derivatives and multiplication with the corresponding functions we obtain

$$\begin{pmatrix} \hat{a} & 0 \\ \hat{c} & \hat{d} \end{pmatrix} \in \Sigma^{(0,0)},$$

which means that this matrix is bounded. Hence, by setting

$$\gamma_0 := \sup_{(t,x,\xi)\in\Upsilon\times\dot{\mathbb{R}}^d} \sup_{|(u,v)|=1} \left(\langle \hat{a}u, u\rangle + \mathrm{Re}\langle \hat{c}u, v\rangle + \langle \hat{d}v, v\rangle \right),$$

we obtain

$$\mathrm{Im}\, \Phi(\mathscr{A}_{00}, \mathscr{A}_{01}, I, 0) \le \frac{\gamma_0}{2r^2} I,$$

as desired, because of $\mathrm{Im}\, \Phi(A_{00}, A_{01}, I, 0) \le 0$. □

Step 3: Existence and Uniqueness. For all $U_0 \in L^2$, $F \in L^1L^2$, the system (1.1) possesses a unique solution $U \in \mathscr{C}L^2$.

Proof. We will just treat the L^2 case, the other cases follow from the previous reductions. The uniqueness comes with the linearity of the problem and the energy estimate. If U and $\widetilde{U}$ are two different solution to (1.1), then the difference function $V = U - \widetilde{U}$ solves the problem

$$\begin{cases} D_t V = A(t, x, D_x)V, & \text{in } (0,T) \times \mathbb{R}^d, \\ V(0,x) \equiv 0, & \text{on } \mathbb{R}^d. \end{cases}$$

This solution has to satisfy the energy estimate with initial data $V_0 \equiv 0$ and $F \equiv 0$. Thus, $\|V\|_{\mathscr{C}L^2} \lesssim 0$, which means $V \equiv 0$ and $U = \widetilde{U}$.

We now will use an duality argument to show the existence of a solution. For the operator $L = D_t - A$ we get from Theorem 4.5 the energy estimate

$$\begin{aligned}\|U(t,\cdot)\|_{L^2(\mathbb{R}^d)} &\lesssim \|U_0\|_{L^2(\mathbb{R}^d)} + \int_0^t \|(LU)(t',\cdot)\|_{L^2(\mathbb{R}^d)}\,dt' \\ &\lesssim \|U_0\|_{L^2(\mathbb{R}^d)} + \sqrt{t}\left(\int_0^t \|(LU)(t',\cdot)\|^2_{L^2(\mathbb{R}^d)}\,dt'\right)^{1/2}.\end{aligned}$$

Squaring leads to

$$\|U(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \lesssim \|U_0\|^2_{L^2(\mathbb{R}^d)} + t\int_0^t \|(LU)(t',\cdot)\|^2_{L^2(\mathbb{R}^d)}\,dt'$$

for all $0 \le t \le T$, especially

$$\|U(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \lesssim_T \|U_0\|^2_{L^2(\mathbb{R}^d)} + \int_0^T \|(LU)(t',\cdot)\|^2_{L^2(\mathbb{R}^d)}\,dt'.$$

By reversing time, that is changing t to $T-t$, we also obtain

$$\|U(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \lesssim \|U(T,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \int_0^T \|(LU)(t',\cdot)\|^2_{L^2(\mathbb{R}^d)}\,dt'. \tag{4.3}$$

The energy estimate (4.3) can then be applied to the adjoint operator as $L^* = -D_t - A^*$. We now introduce the space

$$\mathscr{T} := \{\varphi \in \mathscr{C}^\infty([0,T],\mathscr{S}(\mathbb{R}^d)) \mid \varphi(T) = 0\}.$$

Applying the a priori estimate to functions in $\mathscr{T}$, we have

$$\|\varphi(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \lesssim \int_0^T \|(L^*\varphi)(t',\cdot)\|^2_{L^2(\mathbb{R}^d)}\,dt'$$

for all $\varphi \in \mathscr{T}$ and $t \in [0,T]$ and thus $\|\varphi(t,\cdot)\|_{L^2(\mathbb{R}^d)} \lesssim \|(L^*\varphi)\|_{L^2((0,T),L^2(\mathbb{R}^d))}$. Hence, the operator L^* restricted to $\mathscr{T}$ is one-to-one. So, we can define a unique linear map B on $L^*\mathscr{T}$ via

$$B(L^*\varphi) = \int_0^T \langle F(t),(L^*\varphi)(t)\rangle_{L^2(\mathbb{R}^d)} + \langle U_0,\varphi(0)\rangle_{L^2(\mathbb{R}^d)}.$$

By using the Cauchy-Schwarz-inequality, we derive

$$\begin{aligned}|B(L^*\varphi)| &\le \int_0^T \|F(t)\|_{L^2(\mathbb{R}^d)}\|\varphi(t)\|_{L^2(\mathbb{R}^d)}\,dt + \|U_0\|_{L^2(\mathbb{R}^d)}\|\varphi(0)\|_{L^2(\mathbb{R}^d)} \\ &\le \|L^*\varphi\|_{L^2((0,T),L^2(\mathbb{R}^d))}\int_0^T \|F(t)\|_{L^2(\mathbb{R}^d)}\,dt \\ &\qquad + \|U_0\|_{L^2(\mathbb{R}^d)}\|L^*\varphi\|_{L^2((0,T),L^2(\mathbb{R}^d))} \\ &= \left(\|F\|_{L^1L^2} + \|U_0\|_{L^2(\mathbb{R}^d)}\right)\|L^*\varphi\|_{L^2((0,T),L^2(\mathbb{R}^d))}.\end{aligned}$$

By the Hahn-Banach theorem, B extends to a continuous form on $L^2((0,T), L^2(\mathbb{R}^d))$. Moreover, by Riesz' representation theorem, there exists a unique function $U \in L^2L^2$ such that

$$B(L^*\varphi) = \int_0^T \langle U(t), (L^*\varphi)(t)\rangle_{L^2(\mathbb{R}^d)}\, dt$$

for all $\varphi \in \mathscr{T}$. Additionally for $\varphi \in \mathscr{D}((0,T)\times\mathbb{R}^d)$, we have

$$\int_0^T \langle F(t), \varphi(t)\rangle_{L^2(\mathbb{R}^d)} = \int_0^T \langle (LU)(t), \varphi(t)\rangle_{L^2(\mathbb{R}^d)}$$

This means, that we have $LU = F$ in sense of distributions. By the differential equation, it is $\partial_t U \in L^2((0,T), H^{-1}(\mathbb{R}^d))$ and so $U \in \mathscr{C}([0,T], H^{-1}(\mathbb{R}^d))$. A further integration also gives

$$\langle U_0, \varphi_0\rangle_{L^2(\mathbb{R}^d)} = \langle U(0), \varphi(0)\rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)}.$$

By standard arguments it is $U(0) = U_0$. We now have to control the regularity. We note that, if F and U_0 are smooth, say $F \in \mathscr{C}^\infty([0,T], \mathscr{S}(\mathbb{R}^d))$ and $U_0 \in \mathscr{S}(\mathbb{R}^d)$, the previous construction leads to $U \in \mathscr{C}^\infty([0,T], \mathscr{S}(\mathbb{R}^d))$. If they have less regularity, we have to apply mollifiers. Therefore, let $\{F_k\}_{k\in\mathbb{N}}$ and $\{G_k\}_{k\in\mathbb{N}}$ sequences in $\mathscr{C}^\infty([0,T], \mathscr{S}(\mathbb{R}^d))$ and $\mathscr{S}(\mathbb{R}^d)$, respectively, with

$$\begin{aligned} F_k &\overset{k\to\infty}{\longrightarrow} F \ \text{ in } L^1L^2 \\ G_k &\overset{k\to\infty}{\longrightarrow} U_0 \ \text{ in } L^2. \end{aligned}$$

For all $k \in \mathbb{N}$, there are solutions $U_k \in C^\infty([0,T], \mathscr{S}(\mathbb{R}^d))$ to the source terms F_k and G_k. Applying the energy estimate, we get for arbitrary U_k and U_m

$$\|U_k - U_m\|_{\mathscr{C}L^2} \lesssim \|G_k - G_m\|_{L^2} + \|F_k - F_m\|_{L^1L^2}.$$

Thus, $\{U_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{C}L^2$, say with limit $\widetilde{U} \in \mathscr{C}L^2$. When passing to the limit in the energy estimate, we also see $L\widetilde{U} = F$ and $\widetilde{U}(0) = U_0$. Uniqueness gives us $\widetilde{U} = U$. $\square$

4.4 Higher-order scalar equation

We end this Chapter with an application to higher-order scalar equations.

Let us consider the following m-th order scalar equation

$$\begin{cases} Pu = f(t,x), & (t,x) \in (0,T)\times\mathbb{R}^d, \\ (D_t^j u)(0,x) = u_j(x), & 0 \le j \le m-1, \end{cases}$$

where the operator P is given by

$$P = D_t^m + \sum_{j=1}^{m} a_j(t,x,D_x)D_t^{m-j} \tag{4.4}$$

with coefficients $a_j(t,x,D_x) \in \mathrm{Op}(\widetilde{\Sigma}^{j,\gamma j})$, $\gamma \in \mathbb{Z}$ and $\gamma \leq 2$. In case $m = 2$, $\gamma = 0$, we have our model problem, when $a_1 \equiv 0$ and

$$a_2 = a_2(t,x,D_x) = \sigma^2 \Delta_x \in \mathrm{Op}(\widetilde{\Sigma}^{2,0}).$$

We assume P to be strictly hyperbolic in the sense that

$$\sigma_\Psi^m(P) = \prod_{k=1}^{m} \left(\tau - \lambda_k(t,x,\xi)\right),$$

where $\lambda_k \in \Sigma^{(1),\gamma}$ and

$$|\lambda_k(t,x,\xi) - \lambda_l(t,x,\xi)| \gtrsim \sigma^{1-\gamma}|\xi|$$

for $k \neq l$. We now convert (4.4) into a $m \times m$ system of the first order. The scalar Cauchy problem is then equivalent to the Cauchy problem

$$\begin{cases} D_t U = A(t,x,D_x)U + F(t,x), & (t,x) \in (0,T) \times \mathbb{R}^d, \\ U(0,x) = U_0(x), \end{cases}$$

where

$$U = (\Lambda^{m-1,\gamma(m-1)}u, \Lambda^{m-2,\gamma(m-2)}D_t u, \Lambda^{m-3,\gamma(m-3)}D_t^2 u, \ldots, D_t^{m-1}u)^T.$$

In case $\gamma = 2$, the matrix A is given by the operator-valued matrix

$$A = \begin{pmatrix} A_{00} & A_{01} & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & \\ 0 & 0 & \ddots & \ddots & \ddots & A_{m-2,m-2} & A_{m-2,m-1} \\ A_{m-1,0} & A_{m-1,1} & A_{m-1,2} & \cdots & \cdots & A_{m-1,m-2} & A_{m-1,m-1} \end{pmatrix}.$$

with

$$\begin{aligned} A_{jj} &= (D_t \Lambda^{m-1-j,2(m-1-j)})(\Lambda^{m-1-j,2(m-1-j)})^{-1} \in \mathrm{Op}(\widetilde{\Sigma}^{0,2}), \\ A_{m-1,m-1} &= -a_1 \in \mathrm{Op}(\widetilde{\Sigma}^{1,2}), \\ A_{j,j+1} &= \Lambda^{m-1-j,2(m-1-j)}(\Lambda^{m-2-j,2(m-2-j)})^{-1} \in \mathrm{Op}(\widetilde{\Sigma}^{1,2}), \\ A_{m-1,j} &= -a_{m-j}(\Lambda^{m-1-j,2(m-1-j)})^{-1} \in \mathrm{Op}(\widetilde{\Sigma}^{1,2}), \end{aligned}$$

for $0 \leq j \leq m-2$. In fact, all components belong to $\mathrm{Op}(\widetilde{\Sigma}^{1,2})$, as desired for a first-order system. The right hand side F and the initial data read as

$$\begin{aligned} F &= (0, 0, \dots, 0, \Lambda^{0,-2m+2} f)^T \quad \text{and} \\ U_0 &= (\langle D_x \rangle^{-2j} \langle |x| D_x \rangle^{m-1+j} u_j)^T_{0 \leq j \leq m-1}. \end{aligned}$$

An computation of the principal symbol yields

$$\sigma^1_\Psi(A) = \frac{|\xi|}{\sigma} S_0(t,x,\xi) = \frac{|\xi|}{\sigma} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 1 \\ -p_m & -p_{m-1} & -p_{m-2} & \cdots & \cdots & -p_2 & -p_1 \end{pmatrix},$$

where

$$p_{m-j}(t,x,\xi) = \left(\frac{|\xi|}{\sigma} \right)^{-m+j} \sigma^{m-j}_\Psi(a_{m-j}).$$

For the second symbol we get

$$\sigma^{0,2}_{\Psi,d}(A) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ -q_m & -q_{m-1} & \cdots & -q_2 & -q_1 \end{pmatrix} + \frac{i}{2r^2} \begin{pmatrix} 1-m & 0 & \cdots & 0 \\ 0 & 2-m & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_m & z_{m-1} & \cdots & z_1 \end{pmatrix},$$

where

$$q_{m-j}(t,x,\xi) = \left(\frac{|\xi|}{r} \right)^{-m+1+j} \sigma^{m-j-1,2(m-j-1)}_{\Psi,d}(a_{m-j})$$

and

$$z_{m-j} = \left(\frac{|\xi|}{r} \right)^{-m+1+j} \left(2(m-1-j)(a_{m-j})_{00,x} x - 2(m-1-j)^2 (a_{m-j})_{00} \frac{\langle x, \xi \rangle}{|\xi|^2} \right)$$

for $0 \leq j \leq m-1$. Now, one can provide a symmetrizer M_0 for S_0, namely

$$M_0(t,x,\xi) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_m \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1^{m-1} & \mu_2^{m-1} & \cdots & \mu_m^{m-1} \end{pmatrix},$$

with $\mu_k = \sigma \lambda_k$, similarly as in [**DW1**]. For this matrix we have

$$M_0(t,x,\xi)^{-1} = \prod_{k>l} (\mu_k - \mu_l).$$

One now has to compute both, δ_0 and γ_0. However, the term $\Phi(A_{00}, A_{01}, M_{00}, M_{01})$ becomes quite complicated, so we postpone the study of Φ to future studies.

Let us now return to the model problem where $P = D_t^2 + \sigma^2\Delta_x = D_t^2 + \Lambda^{2,0}$. Then we have to apply the reduction scheme ($m = 2$, $\gamma = 0$) via $U = (\Lambda^{1,0}u, D_t u)^T$, which gives a system of the form (1.1) with

$$A(t,x,D_x) = \begin{pmatrix} (D_t\Lambda^{1,0})(\Lambda^{1,0})^{-1} & \Lambda^{1,0} \\ -\Lambda^{2,0}(\Lambda^{1,0})^{-1} & 0 \end{pmatrix} \in \mathrm{Op}(\widetilde{\Sigma}^{1,2}).$$

For this operator we get in our shortened notation

$$A_0 = \begin{pmatrix} 0 & \sigma|\xi| \\ -\sigma|\xi| & 0 \end{pmatrix} \text{ and } A_{01} = \frac{i}{2r^2}\begin{pmatrix} -I & 0 \\ 0 & 0 \end{pmatrix},$$

while $A_{00} = 0$. Note that A_0 is already Hermitian. Thus, condition (4.1) turns into

$$\operatorname{Im}\Phi(A_{00}, A_{01}, I, 0) = A_{01} \le \frac{\delta_0}{2r^2} I,$$

and is valid for $\delta_0 = 0$. The formula for γ_0 has contributions just from the component $\hat{a} = -I$. Hence, we have to compute the greatest eigenvalue of the matrix

$$\begin{pmatrix} -I & 0 \\ 0 & 0 \end{pmatrix},$$

which is certainly 0. Thus, $\gamma_0 = 0$. Now, if $F = (0, f)^T \in L^1L^2$ and $U_0 = ((\Lambda^{1,0}u)_{|t=0}, (D_t u)_{|t=0})^T \in H^{0,0}$, then the Cauchy problem (1.1) admits an unique solution in $U \in \mathscr{C}L^2$, in view of Theorem 4.2. Transforming back to u gives $u \in \mathscr{C}H^{1,0} \cap \mathscr{C}^1L^2$. Thus, we have proven Theorem 1.2 for $s = 0$. The case of higher regularity comes with the next remark.

Remark 4.11. If $\gamma = 1$, our theory contains strictly hyperbolic equations. For $m = 2$, $a_1 \equiv 0$ and $a_2 = \Lambda^{2,2}$ we get the wave operator $P = D_t^2 + \Delta$. Then we have to apply the reduction scheme via $U = (\Lambda^{1,1}u, D_t u)^T$, which gives a system of the form (1.1) with

$$A(t,x,D_x) = \begin{pmatrix} (D_t\Lambda^{1,1})(\Lambda^{1,1})^{-1} & \Lambda^{1,1} \\ -\Lambda^{2,2}(\Lambda^{1,1})^{-1} & 0 \end{pmatrix} \in \mathrm{Op}(\widetilde{\Sigma}^{1,2}).$$

For this operator we get in our shortened notation

$$A_0 = \begin{pmatrix} 0 & |\xi| \\ -|\xi| & 0 \end{pmatrix} \text{ and } A_{01} = \frac{i}{2r^2}\begin{pmatrix} -I & 0 \\ 0 & 0 \end{pmatrix},$$

while $A_{00} = 0$. Again, A_0 is already Hermitian. Thus, condition (4.1) turns into

$$\operatorname{Im}\Phi(A_{00}, A_{01}, I, 0)) = A_{01} \le \frac{\delta_0}{2r^2} I,$$

and is valid for $\delta_0 = 0$. Similarly as before, $\gamma_0 = 0$. Now, if $F = (0, f)^T \in L^1 L^2$ and $U_0 = ((\Lambda^{1,0} u)_{|t=0}, (D_t u)_{|t=0})^T \in H^{0,0}$, then the Cauchy problem (1.1) admits an unique solution in $U \in \mathscr{C} L^2$, in view of Theorem 4.2. Transforming back to u gives $u \in \mathscr{C} H^{1,1} \cap \mathscr{C}^1 L^2$. Note that $H^{1,1}$ is exactly the classical Sobolev space H^1, so this reproduced the results from the classical strictly hyperbolic theory.

Remark 4.12. Let $\gamma \leq 1$. Transforming the operator P gives $A \in \mathrm{Op}(\widetilde{\Sigma}^{1,\gamma}) + \mathrm{Op}(\widetilde{\Sigma}^{0,2})$ with $A_{00} = 0$, in general. It turns out that, if $u_j \in H^{s+m-j-1, \delta+\gamma(m-j-1)}$ for $0 \leq j \leq m-1$ and $f \in L^1 H^{s,\delta}$, then the Cauchy problem

$$\begin{cases} Pu = f(t,x), & \text{in } (0,T) \times \mathbb{R}^d, \\ (D_t^j u)(0,x) = u_j(x) & \text{on } \mathbb{R}^d, \ \ 0 \leq j \leq m-1, \end{cases}$$

possesses an unique solution

$$u \in \bigcap_{j=0}^{m-1} \mathscr{C}^j H^{s+m-j-1, \delta+\gamma(m-j-1)}.$$

5 Summary and Open Problems

In this thesis we developed a pseudodifferential calculus for a class of degenerate hyperbolic equations. We motivated that question by an problem in fluid dynamics. We introduced amplitude functions and provided a class of pseudodifferential operators that degenerate like $t + |x|^2$ as $(t, x) \longrightarrow (0, 0)$. In $(t, x) = (0, 0)$ these operators are of type $(1, 1)$, but we were able to prove $\mathscr{C}([0, T], L^2(\mathbb{R}^d))$-continuity.

We defined an adapted scale of Sobolev spaces, where at $t = 0$ these spaces are 2-microlocal Sobolev spaces with respect to the Lagrangian $T_0^*\mathbb{R}^d$. With an symbolic approach we derived energy estimates in the spaces and finally proved wellposedness of the corresponding Cauchy problem.

However, there are still open problems, which are devoted to future studies.

In our setting we imposed boundary conditions that are violated in one single point. It would be interesting to concern the case, where these degeneracy happened in a more general compact set.

Our Sobolev spaces are inappropriate to analyze wellposedness in $L^2((0, T), L^2(\mathbb{R}^d))$. Therefore one needs to define that a function u belongs to $\mathscr{H}^{s,\delta}((0, T) \times \mathbb{R}^d)$ if

$$\Lambda^{s-j,\delta-2j} D_t^j u \in L^2((0, T) \times \mathbb{R}^d), \qquad 0 \leq j \leq s.$$

For general $s \in \mathbb{R}$, $\delta \in \mathbb{R}$, the space $\mathscr{H}^{s,\delta}$ is then defined by interpolation and duality. In these spaces, further investigations using functional analysis are desired, in particular a trace theorem.

Another question would be the analysis of the semilinear or even nonlinear problem, which certainly will need paralinearization.

In applications to explicit Cauchy problems, the expression $\Phi(A_{00}, A_{01}, M_{00}, M_{01})$ becomes quite intricate. However, this term is needed to compute both, δ_0 and γ_0. To give better upper or lower bounds, or even an optimal choice for one of these parameters, one has to study $\Phi(A_{00}, A_{01}, M_{00}, M_{01})$ more carefully.

Moreover, one should study the propagation of 2-microlocal singularities and their relation to both, δ_0 and γ_0. We also expect lower bounds on these parameters and it would be interesting to analyze and interpret them in view of a 2-microlocal geometry.

Appendix

A

A.1 Oscillatory Integrals

In this section we will give a short introduction to oscillatory integrals, which is a main ingredient to prove formulas for compositions and adjoints of pseudodifferential operators. The term has been introduced by Lax to deal with integrals of the form

$$Af(x) = \int e^{iS(x,\xi)} a(x,\xi) \hat{f}(\xi)\, đ\xi.$$

We follow the explanations as they are presented in [**Kum**].

Definition A.1 ([**Kum**], Def. 6.1)**.** We say that a $\mathscr{C}^\infty$-function $a(\eta, y)$ in $\mathbb{R}^d_\eta \times \mathbb{R}^d_y$ belongs to $\mathscr{A}^m_{\delta,\tau}$ ($m \in \mathbb{R}$, $0 \leq \delta < 1$, $0 \leq \tau$) if for any $(\alpha, \beta) \in \mathbb{N}_0^{2d}$ there exists a $C_{\alpha,\beta}$ such that

$$|\partial_\eta^\alpha \partial_y^\beta a(\eta, y)| \leq C_{\alpha,\beta} \langle \eta \rangle^{m+\delta|\beta|} \langle y \rangle^\tau.$$

It turns out, that $\mathscr{A}^m_{\delta,\tau}$ is a Fréchet space and one sets

$$\mathscr{A} = \bigcup_{0 \leq \delta < 1} \bigcup_{m \in \mathbb{R}} \bigcup_{0 \leq \tau} A^m_{\delta,\tau}.$$

Definition A.2 ([**Kum**], Def. 6.2)**.** For an element $a(\eta, y)$ of $\mathscr{A}$ we can define the oscillatory integral $\mathrm{Os}(e^{-iy\eta} a)$ by

$$\begin{aligned} \mathrm{Os}(e^{-iy\eta} a) &= \mathrm{Os} - \iint e^{-iy\eta} a(\eta, y) dy đ\eta \\ &= \lim_{\varepsilon \to 0} \iint e^{-iy\eta} \chi(\varepsilon\eta, \varepsilon y) a(\eta, y) dy đ\eta, \end{aligned}$$

where $\chi(\eta, y) \in \mathscr{S}(\mathbb{R}^{2d})$ with $\chi(0,0) = 1$.

The next theorem shows, that $\mathrm{Os}(e^{-iy\eta} a)$ is welldefined. This means that, the limit is independent of the choice of $\chi \in \mathscr{S}$. With Lebesgue's dominated convergence theorem one can prove the following:

Theorem A.3 (**[Kum]**, Thm. 6.4)**.** *For $a \in \mathscr{A}$, the value of the oscillatory integral $\mathrm{Os}(e^{-iy\eta}a)$ is independent of the choice of $\chi \in \mathscr{S}$ satisfying $\chi(0,0) = 1$. When $a \in \mathscr{A}^m_{\delta,\tau}$, taking positive integers l und l', such that*

$$-2l(1-\delta) + m < -d, \ \ -2l' + \tau < -d,$$

we get

$$\left| \langle y \rangle^{-2l'} \langle D_\eta \rangle^{2l'} \left\{ \langle \eta \rangle^{-2l} \langle D_y \rangle^{2l} a(\eta, y) \right\} \right| \in L^1(\mathbb{R}^{2d})$$

and can be written as

$$\mathrm{Os}(e^{-iy\eta}a) = \iint e^{-iy\eta} \langle y \rangle^{-2l'} \langle D_\eta \rangle^{2l'} \left\{ \langle \eta \rangle^{-2l} \langle D_y \rangle^{2l} a(\eta, y) \right\} dy đ\eta.$$

The following properties are helpful for computations:

Theorem A.4 (**[Kum]**, Thm. 6.6 and 6.7)**.** *Let $\{a_j\}_{j\in\mathbb{N}}$ be a bounded set in $\mathscr{A}$. Assume that there exists an $a \in \mathscr{A}$, such that $a_j(\eta, y) \longrightarrow a(\eta, y)$ in $\mathbb{R}^{2d}$ uniformly on any compact set as $j \longrightarrow \infty$. Then we have*

$$\lim_{j\to\infty} \mathrm{Os}(e^{-iy\eta}a_j) = \mathrm{Os}(e^{-iy\eta}a).$$

If $a \in \mathscr{A}$, then for any $\alpha, \beta \in \mathbb{N}_0^d$ we get

$$\mathrm{Os}(e^{-iy\eta}y^\alpha a) = \mathrm{Os}((-D^\alpha_\eta)e^{-iy\eta}a) = \mathrm{Os}(e^{-iy\eta}D^\alpha_\eta a)$$
$$\mathrm{Os}(e^{-iy\eta}\eta^\beta a) = \mathrm{Os}((-D^\beta_y)e^{-iy\eta}a) = \mathrm{Os}(e^{-iy\eta}D^\beta_y a).$$

A.2 Pseudodifferential operators of type (1,1)

Definition A.5 (**[Hor5]**, Chapter 9)**.** A smooth function $a(x, \eta)$ is said to be in the class $S^m_{1,1}$, if

$$|\partial^\alpha_\eta \partial^\beta_x a(x, \eta)| \lesssim_{\alpha\beta} \langle \eta \rangle^{m-|\alpha|+|\beta|}, \qquad \forall x, \eta \in \mathbb{R}^d$$

The set $\mathrm{Op}(S^m_{1,1})$ is the corresponding class of pseudodifferential operators. In 1972, Ching gave an example of type $(1,1)$-operators, which are not L^2-continuous. In **[Hor5]**, Example 9.3.3, we find

Example A.6. Let ϑ be a fixed vector in $\mathbb{R}^d$ and let $A \in \mathscr{C}^\infty_c(\{\eta \,|\, 1 < |\eta| < 2\})$ and set

$$a(x, \eta) = \sum_{\nu=0}^{\infty} e^{-2^\nu ix\vartheta} A\left(\frac{\eta}{2^\nu}\right).$$

Then $a \in S^0_{1,1}$ and $\mathrm{Op}(a)$ is not L^2-continuous.

The difficulties stem from the behavior at the twisted diagonal of $\hat{a}(\xi,\eta) = \mathscr{F}_{x\to\xi}a(x,\eta)$, which is content of the next theorem.

Theorem A.7 ([**Hor5**], Prop. 9.3.1). *Let $a(x,\eta) \in S^m_{1,1}$ and*

$$\hat{a}(\xi,\eta) = 0 \quad \textit{when} \quad |\xi+\eta|+1 < \frac{|\eta|}{B}$$

for some constant B, then $a(x,D) \in \mathscr{L}(H^{s+m}(\mathbb{R}^d), H^s(\mathbb{R}^d))$ for every $s \in \mathbb{R}$, with norm depending on s, B and seminorms of a.

We now want to apply this theorem for arbitrary $a \in S^m_{1,1}$. The idea in [**Hor5**], Chapter 9, is to remove a certain part of the twisted diagonal. Therefore, one introduces a cutoff $\chi \in \mathscr{C}^\infty(\mathbb{R}^{2d})$, such that

(i) $\chi(t\xi, t\eta) = \chi(\xi,\eta)$ if $t \geq 1$ and $|\eta| \geq 2$,

(ii) $\operatorname{supp}(\chi) \subset \{(\xi,\eta) \,|\, |\xi| \leq |\eta|,\ |\eta| \geq 1\}$ and

(iii) $\chi \equiv 1$ in $\{(\xi,\eta) \,|\, 2|\xi| \leq |\eta|,\ |\eta| \geq 2\}$.

Then one is able to define $a_{\chi,\varepsilon}$ via $\hat{a}_{\chi,\varepsilon}(\xi,\eta) = \chi(\xi+\eta, \varepsilon\eta)\hat{a}(\xi,\eta)$. The function $a_{\chi,\varepsilon}$ has the following properties:

Lemma A.8 ([**Hor5**], Lem. 9.3.2). *Let $a \in S^m_{1,1}$ and $0 < \varepsilon \leq 1$. Then the function $a_{\chi,\varepsilon}$ belongs to $\mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and*

(i) $|\partial_x^\beta \partial_\eta^\alpha a_{\chi,\varepsilon}(x,\eta)| \leq C_{\alpha\beta}\varepsilon^{-|\alpha|}\langle\eta\rangle^{m+|\beta|-|\alpha|}$,

(ii) $\left(\displaystyle\int_{R\leq|\eta|\leq 2R} |\partial_\eta^\alpha a_{\chi,\varepsilon}(x,\eta)|^2\, d\eta\right)^{\frac{1}{2}} \leq C_\alpha R^m (\varepsilon R)^{\frac{d}{2}-|\alpha|}$.

If now $a \in S^m_{1,1}$, then by the previous theorem we obtain

$$a(x,D) - a_{\chi,1}(x,D) \in \mathscr{L}(H^{s+m}(\mathbb{R}^d), H^s(\mathbb{R}^d))$$

for every $s \in \mathbb{R}$. The next step is then to prove H^s-continuity of $a_{\chi,1}(x,D)$ unter suitable assumptions on the behavior at the twisted diagonal of $\hat{a}(\xi,\eta)$.

Theorem A.9 ([**Hor5**], Thm. 9.3.5). *Let $a \in S^m_{1,1}$ and assume that for some $\sigma \in \mathbb{R}_+$ the estimate*

$$\left(\int_{R\leq|\eta|\leq 2R} |\partial_\eta^\alpha a_{\chi,\varepsilon}(x,\eta)|^2\, d\eta\right)^{\frac{1}{2}} \leq C_\alpha \varepsilon^\sigma R^m (\varepsilon R)^{\frac{d}{2}-|\alpha|}, \ R > 0$$

is valid for all $\varepsilon \in (0,1)$ *and all* $\alpha \in \mathbb{N}_0^d$*. Then* $a(x,D) \in \mathscr{L}(H^{s+m}(\mathbb{R}^d), H^s(\mathbb{R}^d))$*for* $s+\sigma > 0$.

As an immediate consequence we get the well-known result from Stein of the form:

Corollary A.10 ([**Hor5**], Cor. 9.3.6). *If* $a \in S^m_{1,1}$ *then* $a(x,D) \in \mathscr{L}(H^{s+m}(\mathbb{R}^d), H^s(\mathbb{R}^d))$ *for every* $s > 0$.

The next result gives characterizations under which conditions an operator of type (1,1) is L^2-continuous.

Theorem A.11 ([**Hor5**], Thm. 9.4.2). *If* $a \in S^m_{1,1}$ *then the following three conditions are equivalent:*

(i) $a(x,D_x)^\dagger \in \mathrm{Op}(S^m_{1,1})$.

(ii) With $a_{\chi,\varepsilon}$ *defined as above there is an estimate*

$$|\partial_\eta^\alpha \partial_x^\beta a_{\chi,\varepsilon}(x,\eta)| \lesssim_{\alpha\beta N} \varepsilon^N \langle\eta\rangle^{m-|\alpha|+|\beta|}, \quad 0<\varepsilon<1,$$

for arbitrary N, α, β.

(iii) $a(x,D_x) \in \mathscr{L}(H^{s+m}(\mathbb{R}^d), H^s(\mathbb{R}^d))$ *for every* $s \in \mathbb{R}$.

Finally one has

$$\Psi^{m,\dagger}_{1,1} := \mathrm{Op}\, S^m_{1,1} \cap (\mathrm{Op}\, S^m_{1,1})^\dagger \subset \mathscr{L}(H^{s+m}(\mathbb{R}^d), H^s(\mathbb{R}^d))$$

for every $s \in \mathbb{R}$.

A.3 2-microlocal Sobolev spaces

In this section we will present some results on second microlocalization, which is microlocalization along a Lagrangian submanifold Z of $T^*\mathbb{R}^d$. We follow the statements in [**Bon**]. Note that the scaling of the symbol spaces is slightly different to those in Chapter 3.

Bony constructed a symbolic calculus, when Z is the conormal of 0 in $\mathbb{R}^d$. A Lagrangian submanifold of $T^*\mathbb{R}^d$ can be defined by d equations

$$m_1(x,\xi) = \ldots = m_d(x,\xi) = 0,$$

where m_j are homogeneous of degree 1, such that the Poisson brackets $\{m_i, m_j\}$ vanish on Z. With M_j we denote the corresponding pseudodifferential operator with principal symbol m_j.

Definition A.12 ([**Bon**], Def. 1.2)**.** For $(s,k) \in \mathbb{R} \times \mathbb{N}_0$ we say that $u \in H^{s,k}_{Z,\mathrm{loc}}$ if $M^I u \in H^s_{\mathrm{loc}}$ for $|I| \leq k$. For $(s,s') \in \mathbb{R}^2$ the spaces $H^{s,s'}_{Z,\mathrm{loc}}$ are defined by duality and interpolation.

Definition A.13 ([**Bon**], Def. 1.3)**.** The space $\Sigma^{m,p}_Z$ consists of all $a(x,\xi) \in \mathscr{C}^\infty(\mathbb{R}^{2d})$ such that

$$|H^I K^J a(x,\xi)| \lesssim \langle \xi \rangle^{m+|I|} \left(1 + \sum |m_j(x,\xi)|\right)^{\frac{p-|I|}{2}}$$

for finite families of vector fields $H_i \in \mathscr{H}_0$ (homogeneous of degree 0) and $K_j \in \mathscr{H}_1$ (vector fields in $\mathscr{H}_0$ and tangent to Z).

In the case of $Z = T^*_0\mathbb{R}^d$, we are able to provide an exact characterization of the 2-microlocal Sobolev spaces. This uses Littlewood-Paley decomposition.

Theorem A.14 ([**Bon**], Thm. 2.4)**.** *A distribution u belongs to $H^{s,s'}_0$ if and only if one has*

$$|2^{js}(1+2^j|x|)^{s'} u_j(x)|_0 \leq c_j, \qquad \sum c_j^2 < \infty.$$

In the case $s' = k \in \mathbb{N}_0$, it is then $u \in H^{s,k}_0 \iff x^\alpha u \in H^{s+|\alpha|}$ for $|\alpha| \leq k$.

Additionally weighted Sobolev spaces are introduced. These spaces are related to the trace spaces $H^{s,\delta}$ in Chapter 3.

Definition A.15 ([**Bon**], Def. 2.5)**.** Let $v \in \mathscr{D}'(\mathbb{R}^d \setminus 0)$, vanishing outside $B(0,1)$. A distribution v belongs to $SP(s,s')$ if and only if one has

$$|\varphi(x)v(2^{-j}x)|_{s+s'} \leq c_j 2^{-j(s-d/2)}, \qquad \sum c_j^2 < \infty.$$

In the case $s+s' \in \mathbb{N}_0$ it is to $|x|^{-s+|\alpha|}D^\alpha v \in L^2(\mathbb{R}^d)$ for $0 \leq |\alpha| \leq s+s'$.

For $u \in \mathscr{D}'(\mathbb{R}^d)$ define

$$\Pi u = \sum_{p \geq q} \varphi(2^p x)\varphi(2^{-q}D_x)u,$$

and for $v \in \bigcup SP(s,s')$ set

$$Pv = \sum_{p \geq q} \varphi(2^{-q}D_x)\varphi(2^p x)u,$$

respectively. Then we get the following mapping properties:

Theorem A.16 ([**Bon**], Def. 2.7)**.** *For $s \geq 0$, $s+s' \geq 0$, $H^{s,s'}_0$ and $SP(s,s')$ are subspaces of $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d \setminus 0)$ and one has* *(i)* $SP(s,s') \subset H_0^{s,s'}$,

(ii) $\Pi\colon H_0^{s,s'} \longrightarrow SP(s,s')$,

(iii) $P\colon SP(s,s') \longrightarrow H_0^{s,s'}$.

The results (ii) and (iii) are also valid for $(s,s') \in \mathbb{R}^2$.

In the case $Z = T_0^*\mathbb{R}^d$ we get from Definition A.13

$$a(x,\xi) \in \Sigma_0^{m,p} \iff |D_\xi^\alpha D_x^\beta a(x,\xi)| \lesssim \langle\xi\rangle^{m-|\alpha|+|\beta|}(1+|x||\xi|)^{p-|\beta|}. \tag{A.1}$$

To those symbols one can associate operators defined on the spaces $H_0^{s,s'}$. Therefore, one uses operators defined on $SP(s,s')$ and isomorphisms.

Definition A.17 ([**Bon**], Def. 3.2)**.** A smooth function $a(x,\xi)$ defined for $x \neq 0$ belongs to $S\Sigma_0^{m,p}$, if

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \lesssim |x|^{-m+|\alpha|-|\beta|}(1+|x||\xi|)^{m+p-|\alpha|}$$

and

$$\operatorname{supp}(\hat{a}^2(x,x-y)) \subset \left\{(x,y) \mid \frac{1}{k} \leq \frac{|y|}{|x|} < k\right\}$$

for some $k > 1$. Here $\hat{a}^2$ means the Fourier transform with respect to ξ.

This two different worlds are now connected by the following theorem:

Theorem A.18 ([**Bon**], Thm. 3.3)**.** *Let χ be a smooth function on $\mathbb{R}$, equal to 1 for $|t| \leq 1/4$ and 0 for $|t| \geq 1/2$. Then, the map*

$$a(x,\xi) \longmapsto (1-\chi(|x||\xi|))a(x,\xi)$$

induces isomorphisms (independent of χ):

$$S\Sigma_0^{m,p}/S\Sigma_0^{m,-\infty} \xrightarrow{\cong} \Sigma_0^{m,p}/\Sigma_0^{m,-\infty}.$$

Finally, the following mapping theorem turns out:

Theorem A.19 ([**Bon**], Thm. 3.6)**.** *For $(s,s'),(m,p) \in \mathbb{R}^2$ it is*

$$\operatorname{Op}(S\Sigma^{m,p})\colon SP(s,s') \longrightarrow SP(s-m,s'-p).$$

A.4 Asymptotic sums

We briefly present a result, which is a general scheme to carry out asymptotic sums. This scheme was used to prove asymptotic completeness, see Proposition 3.8. A proof can be found in [**Sch1**].

Proposition A.20 ([**Sch1**], Prop. 1.1.17). *Let E^j, $j \in \mathbb{N}$, be a sequence of Fréchet spaces with continuous embeddings $E^{j+1} \hookrightarrow E^j$ for all j. Set $E^\infty = \varprojlim_{j\in\mathbb{N}} E^j$. Assume that there exists a c-dependent family of linear operators*

$$\chi^j(c) \colon E^j \longrightarrow E^j$$

for all $j \in \mathbb{N}$, $c \in \mathbb{R}_+$, with the following properties:

(i) We have for every $j \in \mathbb{N}$:

$$e - \chi^j(c)e \in E^\infty$$

for all $c \in \mathbb{R}_+$, $e \in E^j$.

(ii) The diagram

$$\begin{tikzcd} E^{j+1} \arrow[r, hook] \arrow[d, "\chi^{j+1}(c)"'] & E^j \arrow[d, "\chi^j(c)"] \\ E^{j+1} \arrow[r, hook] & E^j \end{tikzcd}$$

commutes for all $j \in \mathbb{N}$, $c \in \mathbb{R}_+$.

(iii) If $\{r_k^j\}_{k\in\mathbb{N}}$ is a semi-norm system, that defines the Fréchet topology in E^j, then for arbitrary fixed $j, k \in \mathbb{N}$ there exists an $l(j,k) \geq j$, such that $f \in E^m$ for every $m \geq l(j,k)$ implies

$$r_k^j(\chi^m(c)f) \longrightarrow 0$$

as $c \longrightarrow \infty$.

Then, for every sequence $e_j \in E_j$, $j \in \mathbb{N}$, there exists a sequence of constants $c_j \in \mathbb{R}_+$, such that

$$\sum_{j=k}^{\infty} \chi^j(c_j)e_j$$

converges in E^k for every $k \in \mathbb{N}$. In other words

$$e := \sum_{j=0}^{\infty} \chi^j(c_j)e_j$$

converges in E^0 and has the property

$$e - \sum_{j=0}^{N} e_j \in E^{N+1}$$

for all $N \in \mathbb{N}$. Moreover, e is unique modulo E^∞.

A.5 The $T(1)$-Theorem

We first recall some results of the theory of BMO-functions and singular integrals of nonconvolution type. We will follow [**Gra2**].

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $Q \subset \mathbb{R}^d$ be a measurable set. Then we denote by

$$M_Q(f) := \frac{1}{|Q|} \int_Q f(x)\, dx$$

the mean or average of f over Q. Furthermore, we call the function $|f - M_Q(f)|$ the oscillation of f over Q. Then the expression

$$M_Q(|f - M_Q(f)|)$$

can be interpreted as the mean oscillation of f over Q. This leads us now to the following definition

Definition A.21 **([Gra2], Def. 7.1.1).** For f a complex-valued locally integrable function on $\mathbb{R}^d$, set

$$\|f\|_{\text{BMO}} := \sup_Q M_Q(|f - M_Q(f)|),$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^d$. The function f is called of bounded mean oscillation if $\|f\|_{\text{BMO}} < \infty$. Further we set

$$\text{BMO}(\mathbb{R}^d) = \{f \in L^1_{\text{loc}}(\mathbb{R}^d) \mid \|f\|_{\text{BMO}} < \infty\}.$$

It is not hard to see, that $\text{BMO}(\mathbb{R}^d)$ is a linear space. This means, if $f, g \in \text{BMO}(\mathbb{R}^d)$ and $\lambda \in \mathbb{C}$, then $f + g \in \text{BMO}(\mathbb{R}^d)$ and $\lambda f \in \text{BMO}(\mathbb{R}^d)$. Moreover, we get

$$\|f + g\|_{\text{BMO}} \leq \|f\|_{\text{BMO}} + \|g\|_{\text{BMO}}$$
$$\|\lambda f\|_{\text{BMO}} \leq |\lambda| \|f\|_{\text{BMO}}.$$

However, we cannot hope, that $(\text{BMO}(\mathbb{R}^d), \|\cdot\|_{\text{BMO}})$ forms a normed vector space, since $\|c\|_{\text{BMO}} = 0$ for every constant function c. The converse is also true. In fact, one can show, that if $\|f\|_{\mathrm{BMO}} = 0$, then f is almost everywhere equal to a constant. For later use, we will prove the following result:

Lemma A.22 ([**Gra2**], Prop. 7.1.2,(2))**.** *The space* $L^\infty(\mathbb{R}^d)$ *is contained in* $\mathrm{BMO}(\mathbb{R}^d)$ *and*

$$\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{L^\infty}.$$

Proof. Let Q be an arbitrary cube in $\mathbb{R}^d$. Then we have

$$\begin{aligned} M_Q(|f - M_Q(f)|) &= \frac{1}{|Q|}\int_Q |f - M_Q(f)|\,dx \\ &\leq \frac{1}{|Q|}\int_Q |f(x)|\,dx + \frac{1}{|Q|}\int_Q |M_Q(f)|\,dx \\ &= M_Q(|f|) + \frac{|M_Q(f)|}{|Q|}\int_Q 1\,dx \leq 2M_Q(|f|). \end{aligned}$$

Moreover, we easily estimate

$$M_Q(|f|) \leq \frac{1}{|Q|}\int_Q |f(x)|\,dx \leq \frac{\|f\|_{L^\infty}}{|Q|}\int_Q 1\,dx = \|f\|_{L^\infty}.$$

Thus, we get

$$M_Q(|f - M_Q(f)|) \leq 2\|f\|_{L^\infty}.$$

Taking the supremum over all cubes Q, we get exactly $\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{L^\infty}$. □

Remark A.23. In fact, there are reasonably more properties and results of BMO-functions which will not play any role here. We will later see how this class of functions appear in the concept of proving the $\mathscr{C}L^2$-continuity with help of the $T(1)$-theorem.

We now turn to standard kernels and operators associated to them. Let $\Delta := \{(x,x)\,|\,x \in \mathbb{R}^d\}$.

Definition A.24 ([**Gra2**], Def. 8.1.2)**.** A function K on $(\mathbb{R}^d \times \mathbb{R}^d)\setminus\Delta$, which satisfies for some $B > 0$ the size condition

$$|K(x,y)| \leq \frac{B}{|x-y|^d} \tag{A.2}$$

and for some $\kappa > 0$ the regularity conditions

$$|K(x,y) - K(x',y)| \leq \frac{B|x-x'|^\kappa}{(|x-y| + |x'-y|)^{d+\kappa}} \tag{A.3}$$

whenever $|x - x'| \leq \frac{1}{2}\max(|x-y|, |x'-y|)$ and

$$|K(x,y') - K(x,y')| \leq \frac{B|y-y'|^\kappa}{(|x-y| + |x-y'|)^{d+\kappa}} \tag{A.4}$$

whenever $|y-y'| \leq \frac{1}{2}\max(|x-y|,|x-y'|)$ is called a standard kernel with constants B,κ. The class of all standard kernels with constants B,κ is denoted by $SK(\kappa,B)$.

Let us present some important observations and examples.

Example A.25.

(i) If $K \in SK(\kappa,B)$, then the adjoint kernel $K^* \in SK(\kappa,B)$.

(ii) The kernel $K(x,y)=|x-y|^{-d}$ defined away from Δ belongs to $SK(1,d4^{d+1})$.

(iii) Assume that (A.2) holds and let further

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \leq \frac{B}{|x-y|^{d+1}}$$

for all $x \neq y$, then $K \in SK(1,4^{d+1}B)$.

After introducing the class of standard kernels, we are now able to define linear operators associated to them.

Definition A.26 ([**Gra2**], Def. 8.1.8)**.** Let $0<\kappa,B<\infty$ and $K \in SK(\kappa,B)$. A continuous linear operator $T\colon \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}'(\mathbb{R}^d)$ is said to be associated with K if it satisfies

$$T(f)(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,dy$$

for all $f \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ and $x \notin \operatorname{supp}(f)$. If T is associated with K, then the Schwartz kernel W of T coincides with K on $(\mathbb{R}^d \times \mathbb{R}^d)\setminus\Delta$.

We are now interested in defining the action of such operators on bounded and smooth functions. Therefore, we first define

$$\mathscr{D}_0(\mathbb{R}^d) = \left\{\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} \varphi(x)\,dx = 0\right\}.$$

The space $\mathscr{D}_0(\mathbb{R}^d)$ is equipped with the same topology as the space $\mathscr{D}(\mathbb{R}^d)$. Note that $\mathscr{D}_0'(\mathbb{R}^d) \supseteq \mathscr{D}'(\mathbb{R}^d)$ and $\mathrm{BMO}(\mathbb{R}^d) \subseteq \mathscr{D}_0'(\mathbb{R}^d)$.

Definition A.27 ([**Gra2**], Def. 8.1.16)**.** Let T be a continuous linear operator from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$, that satisfies

$$\langle T(f),\varphi\rangle = \langle W, f\otimes\varphi\rangle$$

for all $f,\varphi \in \mathscr{S}(\mathbb{R}^d)$ and some distribution $W \in \mathscr{S}(\mathbb{R}^{2d})$, that coincides with a standard kernel. Let further be g bounded and smooth and $\eta \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ with

$0 \leq \eta \leq 1$ and equal to 1 in a neighborhood of the support of a given $\varphi \in \mathscr{D}_0(\mathbb{R}^d)$. Then we define $T(f) \in \mathscr{D}_0'(\mathbb{R}^d)$ as

$$\langle T(f), \varphi\rangle = \langle T(f\eta), \varphi\rangle + \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} K(x,y)\varphi(x)\, dx \right) f(y)(1-\eta(y))\, dy.$$

[**Gra2**], Chapter 8.1.3, shows that this definition makes sense as the double integral is an absolutely convergent integral. It is also shown, that this definition of $T(f)$ is independent of the choice of the cut-off function η. To state the $T(1)$-theorem, we need two more definitions.

Definition A.28 ([**Gra2**], Def. 8.3.1)**.** A normalized bump is a smooth function φ supported in the ball $B(0,10)$ that satisfies

$$|(\partial_x^\alpha \varphi)(x)| \leq 1$$

for all multi-indices $|\alpha| \leq 2\left[\frac{d}{2}\right] + 2$, where $[z]$ denotes the integer part of z.

Given a function f on $\mathbb{R}^d$, $R > 0$ and $x_0 \in \mathbb{R}^d$ set

$$\tau^{x_0}(f_R)(y) := R^{-d} f(R^{-1}(y - x_0)).$$

Definition A.29 ([**Gra2**], Def. 8.3.2)**.** We say that a continuous linear operator $T\colon \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}'(\mathbb{R}^d)$ satisfies the weak boundedness property (WBP), if there is a constant C, such that for all f and g normalized bumps, all $x_0 \in \mathbb{R}^d$ and a $R > 0$ we have

$$|\langle T(\tau^{x_0}(f_R)), \tau^{x_0}(g_R)\rangle| \leq CR^{-d}.$$

The smallest constant C is denoted by $\|T\|_{\mathrm{WB}}$.

We are now ready to state the $T(1)$-theorem, which is one the most important ingredients of the sequel analysis. This theorem gives necessary and sufficient conditions for linear operators T with standard kernels to be bounded on $L^2(\mathbb{R}^d)$. The name of theorem $T(1)$ is due to the fact that one of the many equivalent conditions is expressed in terms of properties of the distribution $T(1)$, which can be handled in view of Definition A.27.

Proposition A.30 ([**Gra2**], Theorem 8.3.3)**.** *Let $T\colon \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathscr{S}'(\mathbb{R}^d)$ linear and continuous and let its Schwartz kernel coincides with a standard kernel $K \in SK(\kappa, B)$. Then the following are equivalent:*

(i) It is

$$B_1 = \sup_{\varphi} \sup_{x_0 \in \mathbb{R}^d} \sup_{R>0} R^{d/2} \left(\|T(\tau^{x_0}(f_R))\|_{L^2} + \|T^\dagger(\tau^{x_0}(f_R))\|_{L^2} \right) < \infty,$$

where the first supremum is taken over all normalized bumps φ.

(ii) The operator T satisfies the weak boundedness property and the distributions $T(1)$ and $T^\dagger(1)$ coincide with BMO *functions, that is,*

$$B_2 = \|T(1)\|_{\mathrm{BMO}} + \|T^\dagger(1)\|_{\mathrm{BMO}} + \|T\|_{WB} < \infty.$$

(iii) For every $\xi \in \mathbb{R}^d$ the distributions $T(e^{i\xi\cdot})$ and $T^\dagger(e^{i\xi\cdot})$ coincide with BMO *functions such that*

$$B_3 = \sup_{\xi\in\mathbb{R}^d} \|T(e^{i\xi\cdot})\|_{\mathrm{BMO}} + \sup_{\xi\in\mathbb{R}^d} \|T^\dagger(e^{i\xi\cdot})\|_{\mathrm{BMO}} < \infty.$$

(iv) It is

$$B_4 = \sup_{\varphi} \sup_{x_0\in\mathbb{R}^d} \sup_{R>0} R^d \left(\|T(\tau^{x_0}(f_R))\|_{\mathrm{BMO}} + \|T^\dagger(\tau^{x_0}(f_R))\|_{\mathrm{BMO}} \right) < \infty,$$

where the first supremum is taken over all normalized bumps φ.

(v) T extends to an bounded operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$.

Moreover we have

$$c_{d,\kappa}(B + B_j) \leq \|T\|_{L^2\to L^2} \leq C_{d,\kappa}(B + B_j)$$

for all $j \in \{1,2,3,4\}$ and for some constants $c_{d,\kappa}$, $C_{d,\kappa}$ that depend only on the dimension d and parameter $\kappa > 0$.

[**Gra2**], Chapter 8.3.2, is devoted to the very long proof of this proposition. In fact, there are two more equivalent conditions for the extensions to $L^2(\mathbb{R}^d)$ using truncated operators

$$T^{(\varepsilon)}(f)(x) = \int_{\mathbb{R}^d} K(x,y)\chi_{|x-y|>\varepsilon} f(y)\,dy$$

for $\varepsilon > 0$, see [**Gra2**], Def. 8.1.10.

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Curriculum Vitæ

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Berufliche Laufbahn

10/2012 – heute **Wissenschaftlicher Mitarbeiter (TV-L E13, 100%)**, *Mathematisches Institut*, Georg-August-Universität Göttingen.

- Planung und Durchführung von Lehrveranstaltungen
- Beisitzen von Prüfungen (Bachelor, Master, Diplom)
- Forschung im Rahmen der Dissertation (Abgabe Frühjahr 2018)
- Mitglied im Institutsvorstand als Mittelbauvertreter
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Ausbildung

10/2012 – derzeit	**Promotionsstudium im Fach Mathematik**, *Georg-August-Universität Göttingen*, Betreuer: Prof. Dr. Ingo Frank Witt.
10/2010 – 09/2012	**Masterstudium im Fach Mathematik**, *Universität Rostock*, Rostock, *M.Sc.* Note 1.4, sehr gut
10/2007 – 09/2010	**Bachelorstudium im Fach Mathematik**, *Universität Rostock*, Rostock, *B.Sc.* Note 1.5, sehr gut
09/1999 – 07/2007	**Gymnasium**, *Richard-Wossidlo-Gymnasium*, Ribnitz-Damgarten, *Abitur*. Note 1.7, gut

Auszeichnungen

10/2011-09/2012 Deutschlandstipendium der Universität Rostock

Weiterbildung

02/2017 – 02/2018 **WeWiMento**, *ausgewählter Teilnehmer im universitätsinternen Mentoringprogramm für Wege ins Wissenschaftsmanagement*.

- Teilnahme an Workshops unter anderem zu den Themen Konfliktkompetenz, Souveränes Auftreten, Netzwerken
- Hospitation in der Hochschuldidaktik, in der Führung einer Graduiertenschule sowie in der Abteilung Forschung
- persönliche Begleitung durch einen Mentor

01/2011 – 06/2011 **Ausbildung zum ILIAS-Teletutor**.

- 70 Stunden, Einführung in die Anwendung von ILIAS, einer E-Learning-Plattform für Lehre

04/2013 – 04/2015 **Ausbildung zum nebenamtlichen Chorleiter**.

- Unterricht unter anderem in den Fächern Chorleitung, Tonsatz und Gehörbildung, Partiturspiel, Liturgik, Hymnologie und Geschichte der Kirchenmusik
- Abschluss: C-Chorleiter (Note: 1.0, sehr gut)

Akademische Selbstverwaltung/Service

03/2017 – derzeit Gewähltes Mitglied in der Promovierendenvertretung (PromV) der Universität Göttingen für die Fakultät für Mathematik und Informatik, stellvertretender Sprecher der PromV, beratendes Mitglied des Fakultätsrates

01/2017 – derzeit Vertreter der wissenschaftlichen Mitarbeiter am Mathematischen Institut

10/2012 – derzeit Organisation und Durchführung des jährlich stattfindenden Mathecamps für begabte Schülerinnen und Schüler aus ganz Deutschland am Mathematischen Institut

Ehrenamt

11/2012 – derzeit Mitglied des Kantoreirates der Göttinger Stadtkantorei und Sprecher der Stimmgruppe Tenor

11/2013 – derzeit Mitglied im Beirat des Göttinger Kammerchores

10/2011 Gründungsmitglied und 1. Vorsitzender (bis März 2013) des Freundeskreises der Kirchenmusik in Warnemünde e.V.

Weitere Kenntnisse

Sprachen Deutsch (Muttersprache), Englisch (B2), Russisch (A1)

IT LaTeX, MATLAB, Mathematica, MS Office Anwendungen

Persönliche Interessen

Singen in der Kantorei, Klavier, Orgel und Waldhorn spielen, Kochen, Reisen, Gesellschaftstanz

Publikationen und Vorträge

Publikationen

Frühjahr 2018 M. Krüger: **On the Cauchy problem for a class of degenerate hyperbolic equations**, *Dissertation*, in Vorbereitung

11/2012 J. Merker, M. Krüger: **On a variational principle in thermodynamics** in *Continuum Mech. Thermodyn.*, online als First View on journal page (doi: 10.1007/s00161-012-0277-2) 15 pp.

08/2012 M. Krüger: **Interaktion eines Fluids und eines Körpers: Modellierung und Existenz von schwachen Lösungen**, *Masterthesis* (Universität Rostock)

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Vorträge (Auswahl)

03/2018 **Degenerate hyperbolic equations and their application in fluid dynamics**, *im Rahmen eines dreiwöchigen Forschungsaufenthaltes an der Nanjing Normal University bei dr Arbeitsgruppe von Prof. Yin Huicheng*, China.

11/2017 **Spezielle Zahlenbereiche**, *im Rahmen des „How-to-prove-it"-Wochenendes für Erstsemester, organisiert von der Fachschaft Mathematik*, Göttingen.

03/2016 **On degenerate hyperbolic Cauchy problems**, *International Workshop on Geometric and Singular Analysis*, Potsdam.

09/2015 **Semigroups and their applications to PDEs**, *Summer School on Fourier Integral Operators*, Ouagadougou (Burkina Faso).
Vorlesungsreihe

Konferenzen und Drittmittel

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10/2016 **Conference on Analysis and Colloquium in Honor of the 60th Birthday of Elmar Schrohe**, *Hannover*.

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10/2015 **Workshop on Analysis and PDE**, *Hannover*.

09/2015 **Summer School on Fourier Integral Operators**, *Ouagadougou*.

07/2015 **Summer School on Quantum Ergodicity and Harmonic Analysis, Part IV**, *Marburg*.

02/2015 **International Workshop on Geometric and Singular Analysis**, *Potsdam*.

09/2014 **Summer School on Spectral Geometry**, *Göttingen*.

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03/2014 **International Workshop on Geometric and Singular Analysis**, *Potsdam*.

11/2013 **Winter School on Quantum Ergodicity and Harmonic Analysis, Part III**, *Marburg*.

11/2013 **Joint CRM-ISAAC Conference on Fourier Analysis and Approximation Theory**, *Barcelona*.

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09/2013 **Workshop on Elliptic and Parabolic Equations**, *Hannover*.

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03/2013 **German-Sino Workshop on the Analysis of Partial Differential Equations and Their Applications**, *Göttingen*.

01/2013 **Winter School on Quantum Ergodicity and Harmonic Analysis, Part II**, *Göttingen*.

09/2012 **Summer School on Singular Analysis**, *Oldenburg*.

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12/2017 **DAAD-Reisemittel (Vortragsreisenprogramm)**, *Reisekostenzuschuss für einen Foschungsaufenthalt (März 2018) an der Nanjing Normal University, China*, 850 €.

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Declaration

I hereby declare that this thesis has been composed solely by myself and that it has not been submitted, in whole or in part, in any previous application for a degree. Except where states otherwise by reference or acknowledgment, the work presented is entirely my own.

Göttingen, March 27, 2018

Matthias Krüger

Colophon

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